A COURSE OF MATHEMATICS.

IN THREE VOLUMES.

COMPOSED FOR THE

USE OF THE ROYAL MILITARY ACADEMY.

BY CHARLES HUTTON, LL.D. F.R.S.

LATE PROFESSOR OF MATHEMATICS IN THAT INSTITUTION.

THE EIGHTH EDITION,

WITH MANY CORRECTIONS AND IMPROVEMENTS.

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VOL. II.

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PREFACE

TO THE

NEW EDITION.

In preparing this new edition of the second volume of the *Woolwich Course of Mathematics* for the press, it has been my great object, while I was careful to introduce the principal improvements which the lapse of twenty-five years since its first publication has rendered necessary, to deviate as little as possible from the original plan of my valued friend, the Author. The only essential deviation, therefore, will be found in the department of *Mechanics*, where, instead of intermingling the propositions in Statics and Dynamics, I have classed them separately; and have, as I went along, introduced a few such new demonstrations as this more scientific arrangement naturally required. Under the head of Statics, I have inserted a few propositions on the equilibration of arches; and under that of Dynamics, a useful proposition or two in reference to central forces. Throughout the mechanical portions of the volume I have given to the symbol g, the value $32\frac{1}{6}$ feet, the natural and obvious measure of the force of gravity in our latitude; instead of $16\frac{1}{12}$ feet, the value assigned in former editions of the Course. This change is not a mere matter of fancy; but will be found considerably to facilitate the perusal of all the modern books on Dynamical science, whether published in Great Britain or on the continent.

The table of fluents is greatly enlarged, and the subject of the pressure of earth is introduced as an application of

the method of fluxions, that I might present an investigation founded upon the principles of *Coulomb* and *Prony*. On the cohesive force of timber, iron, and other substances, useful tables and rules have been selected from the treatises of Mr. *Barlow* and Mr. *Tredgold*. And indeed throughout the volume I have interspersed such new observations, demonstrations, and rules, as will, I trust, with the aid of more than 100 additional questions and examples for the exercise of the student, augment the advantages which may accrue from the study of the volume to those who peruse it with a view to the practical applications of mathematics, whether in the military profession, or among surveyors, architects, and civil engineers.

OLINTHUS GREGORY.

Royal Military Academy, Woolwich,
November 1, 1823.

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OF

VOL. II.

APPENDIX.

ERRATA.

Page 16, line 17, *add*, Or still more simply, find 10 + the diff. (log. P — log. B) in the log. tangents. The corresponding log. secant added to log. B = log. H.

19,	9,	*for* sec. A, *read* tan. A + sec. A.
26,	7 b.,	*for* 7916, *read* 7918.
112,	14,	*for* cut DE, *read* bisect DE.
125,	2,	*for* cor. 1, *read* cor. 2.
128,	7 b.,	*for* SC^2, *read* SC^3.
217,	3,	*for* CA and CD, *read* BA and BD.
	4,	*for* B, *read* C.
234,	12 b.,	*for* axlo, *read* axle.
242,		Put the letter B at the bottom of the first diagram.
248,	line 2 b.,	*for* *l*, *read* *b*.
298,	11 b.,	*for* fluxions, *read* fluxion.

A COURSE OF MATHEMATICS, &c.

PLANE TRIGONOMETRY.

DEFINITIONS.

1. Plane Trigonometry treats of the relations and calculations of the sides and angles of plane triangles.

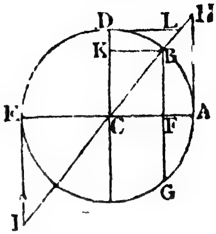
2. The circumference of every circle (as before observed in Geom. Def. 56) is supposed to be divided into 360 equal parts, called Degrees; also each degree into 60 Minutes, and each minute into 60 Seconds, and so on. Hence a semicircle contains 180 degrees, and a quadrant 90 degrees.

3. The Measure of an angle (Def. 57, Geom.) is an arc of any circle contained between the two lines which form that angle, the angular point being the centre; and it is estimated by the number of degrees contained in that arc.

Hence, a right angle, being measured by a quadrant, or quarter of the circle, is an angle of 90 degrees; and the sum of the three angles of every triangle, or two right angles, is equal to 180 degrees. Therefore, in a right-angled triangle, taking one of the acute angles from 90 degrees, leaves the other acute angle; and the sum of the two angles, in any triangle, taken from 180 degrees, leaves the third angle; or one angle being taken from 180 degrees, leaves the sum of the other two angles.

4. Degrees are marked at the top of the figure with a small °, minutes with ′, seconds with ″, and so on. Thus, 57° 30′ 12″, denote 57 degrees 30 minutes and 12 seconds.

5. The Complement of an arc, is what it wants of a quadrant or 90°. Thus, if AD be a quadrant, then BD is the complement of the arc AB; and, reciprocally, AB is the complement of BD. So that, if AB be an arc of 50°, then its complement BD will be 40°.



6. The Supplement of an arc, is what it wants of a semicircle, or 180°. Thus, if ADE be a semicircle, then BDE is the supplement of the arc AB; and, reciprocally, AB is the supplement of the arc BDE. So that, if AB be an arc of 50°, then its supplement BDE will be 130°.

7. The Sine, or Right Sine, of an arc, is the line drawn from one extremity of the arc, perpendicular to the diameter which passes through the other extremity. Thus, BF is the sine of the arc AB, or of the supplemental arc BDE. Hence the sine (BF) is half the chord (BG) of the double arc (BAG).

8. The Versed Sine of an arc, is the part of the diameter intercepted between the arc and its sine. So, AF is the versed sine of the arc AB, and EF the versed sine of the arc EDB.

9. The Tangent of an arc, is a line touching the circle in one extremity of that arc, continued from thence to meet a line drawn from the centre through the other extremity; which last line is called the Secant of the same arc. Thus, AH is the tangent, and CH the secant, of the arc AB. Also, EI is the tangent, and CI the secant, of the supplemental arc BDE. And this latter tangent and secant are equal to the former, but are accounted negative, as being drawn in an opposite or contrary direction to the former.

10. The Cosine, Cotangent, and Cosecant, of an arc, are the sine, tangent, and secant of the complement of that arc, the Co being only a contraction of the word complement. Thus, the arcs AB, BD, being the complements of each other, the sine, tangent, or secant of the one of these, is the cosine, cotangent, or cosecant of the other. So, BF, the sine of AB, is the cosine of BD; and BK, the sine of BD, is the cosine of AB: in like manner, AH, the tangent of AB, is the cotangent of BD; and DL, the tangent of DB, is the cotangent of AB: also, CH, the secant of AB, is the cosecant of BD; and CL, the secant of BD, is the cosecant of AB.

Corol. Hence several important properties easily follow from these definitions; as,

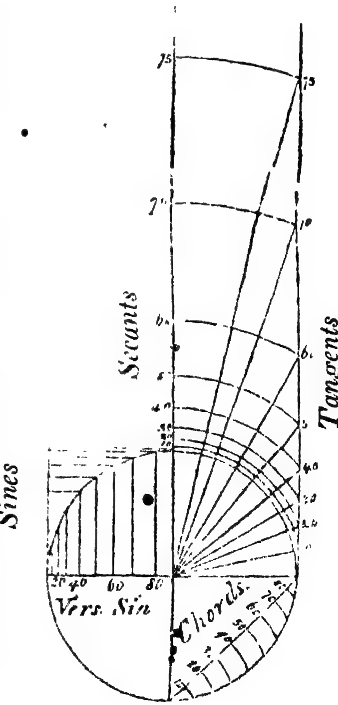
1st, That an arc and its supplement have the same sine, tangent, and secant; but the two latter, the tangent and secant, are accounted negative when the arc is greater than a quadrant or 90 degrees.

2d, When the arc is 0, or nothing, the sine and tangent are nothing, but the secant is then the radius CA, the least it can be. As the arc increases from 0, the sines, tangents, and secants, all proceed increasing, till the arc becomes a whole quadrant AD, and then the sine is the greatest it can be, being the radius CD of the circle; and both the tangent and secant are infinite.

3d, Of any arc AB, the versed sine AF, and cosine BK, or CF, together make up the radius CA of the circle.—The radius CA, the tangent AH, and the secant CH, form a right-angled triangle CAH. So also do the radius, sine, and cosine, form another right-angled triangle CBF or CBK. As also the radius, cotangent, and cosecant, another right-angled triangle CDL. And all these right-angled triangles are similar to each other.

11. The sine, tangent, or secant of an angle, is the sine, tangent, or secant of the arc by which the angle is measured, or of the degrees, &c. in the same arc or angle.

12. The method of constructing the scales of chords, sines, tangents, and secants, usually engraven on instruments, for practice, is exhibited in the annexed figure.



13. A Trigonometrical Canon, is a table showing the length of the sine, tangent, and secant, to every degree and minute of the quadrant, with respect to the radius, which is expressed by unity or 1, with any number of ciphers. The logarithms of these sines, tangents, and secants, are also ranged in the

tables; and these are most commonly used, as they perform the calculations by only addition and subtraction, instead of the multiplication and division by the natural sines, &c, according to the nature of logarithms. Such tables of log. sines and tangents, as well as the logs. of common numbers, greatly facilitate trigonometrical computations, and are now very common. Among the most correct are those published by the author of this Course.

PROBLEM I.

To compute the Natural Sine and Cosine of a Given Arc.

THIS problem is resolved after various ways. One of these is as follows, viz. by means of the ratio between the diameter and circumference of a circle, together with the known series for the sine and cosine, hereafter demonstrated. Thus, the semicircumference of the circle, whose radius is 1, being 3·14159265358979̇3 &c, the proportion will therefore be,

as the number of degrees or minutes in the semicircle,
is to the degrees or minutes in the proposed arc,
so is 3·14159265 &c, to the length of the said arc.

This length of the arc being denoted by the letter a; and its sine and cosine by s and c; then will these two be expressed by the two following series, viz.

$$s = a - \frac{a^3}{2.3} + \frac{a^5}{2.3.4.5} - \frac{a^7}{2.3.4.5.6.7} + \&c.$$

$$= a - \frac{a^3}{6} + \frac{a^5}{120} - \frac{a^7}{5040} + \&c.$$

$$c = 1 - \frac{a^2}{2} + \frac{a^4}{2.3.4} - \frac{a^6}{2.3.4.5.6} + \&c.$$

$$= 1 - \frac{a^2}{2} + \frac{a^4}{24} - \frac{a^6}{720} + \&c.$$

EXAM. 1. If it be required to find the sine and cosine of 1 minute. Then, the number of minutes in 180° being 10800, it will be first, as 10800 : 1 : : 3·14159265 &c. : ·000290888208665 = the length of an arc of one minute. Therefore, in this case,

$$a = \cdot0002908882$$
$$\text{and } \tfrac{1}{6}a^3 = \cdot000000000004 \text{ \&c,}$$
$$\text{the diff. is } s = \cdot0002908882 \text{ the sine of 1 minute.}$$

Also, from 1·
take $\frac{1}{2}a^2 = 0{\cdot}0000000423079$ &c,
leaves $c = {\cdot}9999999577$ the cosine of 1 minute.

Exam. 2. For the sine and cosine of 5 degrees.
Here as $180^\circ : 5^\circ :: 3 \cdot 14159265$ &c. : $\cdot 08726646 = a$ the length of 5 degrees. Hence

$$\begin{aligned} a &= \cdot 08726646 \\ -\tfrac{1}{6}a^3 &= -\cdot 00011076 \\ +\tfrac{1}{120}a^5 &= \cdot 00000004 \end{aligned}$$

these collected give $s = \cdot 08715574$ the sine of 5°.

And, for the cosine,

$$\begin{aligned} 1 &= 1 \cdot \\ -\tfrac{1}{2}a^2 &= -\cdot 00380771 \\ +\tfrac{1}{24}a^4 &= \cdot 00000241 \end{aligned}$$

these collected give $c = \cdot 99619470$ the cosine of 5°.

After the same manner, the sine and cosine of any other arc may be computed. But the greater the arc is, the slower the series will converge, in which case a greater number of terms must be taken, to bring out the conclusion to the same degree of exactness.

Or, having found the sine, the cosine will be found from it, by the property of the right-angled triangle CBF, viz, the cosine CF $= \sqrt{CB^2 - BF^2}$, or $c = \sqrt{1 - s^2}$.

There are also other methods of constructing the canon of sines and cosines, which, for brevity's sake, are here omitted: some of them, however, are explained under the analytical trigonometry in the third volume of this Course.

PROBLEM II.

To compute the Tangents and Secants.

The sines and cosines being known, or found by the foregoing problem; the tangents and secants will be easily found, from the principle of similar triangles, in the following manner:

In the first figure, where, of the arc AB, BF is the sine, CF or BK the cosine, AH the tangent, CH the secant, DL the cotangent, and CL the cosecant, the radius being CA or CB or CD; the three similar triangles CFB, CAH, CDL, give the following proportions:

1*st*, CF : FB :: CA : AH; whence the tangent is known, being a fourth proportional to the cosine, sine, and radius.

2*d*, CF : CB :: CA : CH; whence the secant is known being a third proportional to the cosine and radius: or, being, indeed, the reciprocal of the cosine when the radius is unity.

3d, BF : FC : : CD : DL; whence the cotangent is known, being a fourth proportional to the sine, cosine, and radius.

Or, AH : AC : : CD : DL; whence it appears that the cotangent is a third proportional to the tangent and radius; or the reciprocal of the tangent to radius 1.

4th, BF : BC : : CD : CL; whence the cosecant is known, being a third proportional to the sine and radius; or the reciprocal of the sine to radius 1.

As for the log. sines, tangents, and secants, in the tables, they are only the logarithms of the natural sines, tangents, and secants, calculated as above.

HAVING given an idea of the calculation and use of sines, tangents, and secants, we may now proceed to resolve the several cases of Trigonometry; previous to which, however, it may be proper to add a few preparatory notes and observations, as below.

Note 1. There are three methods of resolving triangles, or the cases of trigonometry; namely, Geometrical Construction, Arithmetical Computation, and Instrumental Operation; of which the first two will here be treated.

In the First Method, The triangle is constructed, by making the parts of the given magnitudes, namely, the sides from a scale of equal parts, and the angles from a scale of chords, or by some other instrument. Then measuring the unknown parts by the same scales or instruments, the solution will be obtained near the truth.

In the Second Method, Having stated the terms of the proportion according to the proper rule or theorem, resolve it like any other proportion, in which a fourth term is to be found from three given terms, by multiplying the second and third together, and dividing the product by the first, in working with the natural numbers; or, in working with the logarithms, add the logs. of the second and third terms together, and from the sum take the log. of the first term; then the natural number answering to the remainder is the fourth term sought.

Note 2. Every triangle has six parts, viz. three sides and three angles. And in every triangle proposed, there must be given three of these parts, to find the other three. Also, of the three parts that are given, one of them at least must be a side; because, with the same angles, the sides may be greater or less in any proportion.

Note 3. All the cases in trigonometry, may be comprised in three varieties only; viz.

1st, When a side and its opposite angle are given.

2d, When two sides and the contained angle are given.
3d, When the three sides are given.

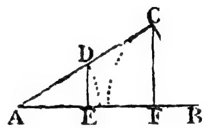
For there cannot possibly be more than these three varieties of cases; for each of which it will therefore be proper to give a separate theorem, as follows:

THEOREM I.

When a Side and its Opposite Angle are two of the Given Parts.

THEN the unknown parts will be found by this theorem; viz. The sides of the triangle have the same proportion to each other, as the sines of their opposite angles have.

That is, As any one side,
Is to the sine of its opposite angle;
So is any other side,
To the sine of its opposite angle.



Demonstr. For, let ABC be the proposed triangle, having AB the greatest side, and BC the least. Take AD = BC, considering it as a radius; and let fall the perpendiculars DE, CF, which will evidently be the sines of the angles A and B, to the radius AD or BC. Now the triangles ADE, ACF, are equiangular; they therefore have their like sides proportional, namely, AC : CF :: AD or BC : DE; that is, the side AC is to the sine of its opposite angle B, as the side BC is to the sine of its opposite angle A.

Note 1. In practice, to find an angle, begin the proportion with a side opposite to a given angle. And to find a side, begin with an angle opposite to a given side.

Note 2. An angle found by this rule is ambiguous, or uncertain whether it be acute or obtuse, unless it be a right angle, or unless its magnitude be such as to prevent the ambiguity; because the sine answers to two angles, which are supplements to each other; and accordingly the geometrical construction forms two triangles with the same parts that are given, as in the example below; and when there is no restriction or limitation included in the question, either of them may be taken. The number of degrees in the table, answering to the sine, measure the acute angle; but if the angle be obtuse, subtract those degrees from 180°, and the remainder will be the obtuse angle. When a given angle is obtuse, or a right one, there can be no ambiguity; for then

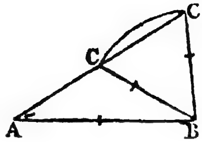
neither of the other angles can be obtuse, and the geometrical construction will form only one triangle.

EXAMPLE I.

In the plane triangle ABC,

Given $\begin{cases} \text{AB } 345 \text{ yards} \\ \text{BC } 232 \text{ yards} \\ \angle \text{A } 37^\circ\ 20' \end{cases}$

Required the other parts.



1. *Geometrically.*

Draw an indefinite line; on which set off AB = 345, from some convenient scale of equal parts.—Make the angle A = $37^\circ\frac{1}{3}$.—With a radius of 232, taken from the same scale of equal parts, and centre B, cross AC in the two points, C, C.—Lastly, join BC, BC, and the figure is constructed, which gives two triangles, and shows that the case is ambiguous.

Then, the sides AC measured by the scale of equal parts, and the angles B and C measured by the line of chords, or other instrument, will be found to be nearly as below; viz.

AC 174	$\angle$ B 27°	$\angle$ C $115^\circ\frac{1}{2}$.
or $374\frac{1}{2}$	or $78\frac{1}{4}$	or $64\frac{1}{2}$.

2. *Arithmetically.*

First, to find the angles at C.

As side	BC	232	- -	log. 2·3654880
To sin. op. $\angle$ A		$37^\circ\ 20'$	- -	9·7827958
So side	AB	345	- -	2·5378191
To sin. op. $\angle$ C		$115^\circ\ 36'$	or $64^\circ\ 24'$	9·9551269
add	$\angle$ A	37 20	37 20	
the sum		152 56	or 101 44	
taken from		180 00	180 00	
leaves	$\angle$ B	27 04	or 78 16	

Then, to find the side AC.

As sine	$\angle$ A	$37^\circ\ 20'$	- -	log. 9·7827958
To op. side	BC	232	- -	2·3654880
So sin.	$\angle$ B	$27^\circ\ 04'$	- -	9·6580371
		78 16	- -	9·9908291
To op. side	AC	174·07	- -	2·2407293
	or	374·56	- -	2·5735213

EXAMPLE II.

In the plane triangle ABC,

Given	AB 365 poles	Ans.	∠ C	98° 3′
	∠ A 57° 12′		AC	154·33
	∠ B 24 45		BC	309·86

Required the other parts.

EXAMPLE III.

In the plane triangle ABC,

Given	AC 120 feet	Ans.	∠ B 64° 34′ 21″
	BC 112 feet		or 115 25 39
	∠ A 57° 27″		∠ C 57 58 39
			or 7 7 21
			AB 112·65 feet
			or 16·47 feet

Required the other parts.

THEOREM II.

When two Sides and their Contained Angle are given.

First find the sum and the difference of the given sides. Next subtract the given angle from 180°, and the remainder will be the sum of the two other angles; then divide that by 2, which will give the half sum of the said unknown angles. Then say,

As the sum of the two given sides,
Is to the difference of the same sides;
So is the tang. of half the sum of their op. angles,
To the tang. of half the diff. of the same angles.

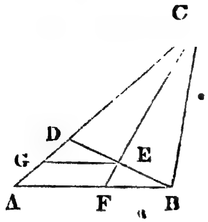
Add the half difference of the angles, so found, to their half sum, and it will give the greater angle, and subtracting the same will leave the less angle: because the half sum of any two quantities, increased by their half difference, gives the greater, and diminished by it gives the less.

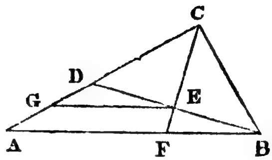
All the angles being thus known, the unknown side will be found by the former theorem.

Note. Instead of the tangent of the half sum of the unknown angles, in the third term of the proportion, may be used the cotangent of half the given angle, which is the same thing.

Demon. Let ABC be a plane triangle, of which AC, CB, and the included angle C are given: C being *acute* in the first figure, *obtuse* in the second.

On AC, the longer side, set off CD = CB the shorter; join





BD, and bisect it in E; also, bisect AD in G, and join GE, CE, producing the latter to F.

Now $\frac{1}{2}$(AC + CB) = $\frac{1}{2}$(2GD + 2DC) = CG
and $\frac{1}{2}$(AC − CB) = $\frac{1}{2}$(2AG) = AG
also $\frac{1}{2}$(A + B) = $\frac{1}{2}$(CDB + CBD) = CBD
and $\frac{1}{2}$(B − A) = ABC − $\frac{1}{2}$ sum = ABD:

also, because CE bisects the base of the isosceles triangle CBD, it is perpendicular to it:

Therefore EC = tangent of CBD } to radius BE.
EF = tangent of ABD }

Lastly, because in the triangle ACF, GE is parallel to AF (Geom. th. 82) we have

CG : GA : : CE : EF; that is,

$\frac{1}{2}$(AC + CB) : $\frac{1}{2}$(AC − CB) : : tan. $\frac{1}{2}$(B + A) : tan. $\frac{1}{2}$(B − A); or, since doubling both the antecedent and consequent of the first ratio does not change the mutual relation of its terms, we have

AC + CB : AC − CB : : tan. $\frac{1}{2}$(B + A) : tan. $\frac{1}{2}$(B − A). Q.E.D.

EXAMPLE I.

In the plane triangle ABC,

Given { AB 345 yards
AC 174·07 yards
∠A 37° 20′ }

Required the other parts.

A

1. *Geometrically.*

Draw AB = 345 from a scale of equal parts. Make the angle A = 37° 20′. Set off AC = 174 by the scale of equal parts. Join BC, and it is done.

Then the other parts being measured, they are found to

be nearly as follow; viz. the side BC 232 yards, the angle B 27°, and the angle C 115°½.

2. *Arithmetically.*

The side AB	345	From	180° 00′
the side AC	174·07	take ∠ A	37 20
their sum	519·07	sum of C and B	142 40
their differ.	170·93	half sum of do.	71 20

As sum of sides AB, AC, - -	519·07	log.	2·7152259
To diff. of sides AB, AC, - -	170·93	-	2·2328183
So tang. half sum ∠s C and B	71° 20	-	10·4712979
To tang. half diff. ∠s C and B	44 16	-	9·9888903
these added give ∠ C	115 36		
and subtr. give ∠ B	27 4		

Then, by the former theorem,

As sin. ∠ C 115° 36′ or 64° 24′ -	log. 9·9551259
To its op. side AB 345 - - -	2·5378191
So sin. of ∠ A 37° 20′ - - -	9·7827958
To its op. side BC 232 - - -	2·3654890

EXAMPLE II.

In the plane triangle ABC,

Given { AB 365 poles; AC 154·33; ∠ A 57° 12′ } Ans. { BC 309·86; ∠ B 24° 45′; ∠ C 98 3 }

Required the other parts.

EXAMPLE III.

In the plane triangle ABC,

Given { AC 120 yards; BC 112 yards; ∠ C 57° 58′ 39″ } Ans. { AB 112·65; ∠ A 57° 27′ 0″; ∠ B 64 34 21 }

Required the other parts.

THEOREM III.

When the Three Sides of a Triangle are given.

First, let fall a perpendicular from the greatest angle on the opposite side, or base, dividing it into two segments, and the whole triangle into two right-angled triangles: then the proportion will be,

As the base, or sum of the segments,
Is to the sum of the other two sides;
So is the difference of those sides,
To the diff. of the segments of the base.

Then take half this difference of the segments, and add it to the half sum, or the half base, for the greater segment and subtract the same for the less segment.

Hence, in each of the two right-angled triangles, there will be known two sides, and the right-angle opposite to one of them; consequently the other angles will be found by the first theorem.

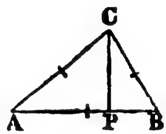
Demonstr. By theor. 35, Geom. the rectangle of the sum and difference of the two sides, is equal to the rectangle of the sum and difference of the two segments. Therefore, by forming the sides of these rectangles into a proportion by theor. 76, Geometry, it will appear that the sums and differences are proportional as in this theorem.

N. B. Before you commence a solution of an example to this case, ascertain whether the triangle be right-angled or not, by determining whether the square of the longest side be equal or unequal to the sums of the squares of the other two. If equal, the example may be referred to the notes to theorem IV.

EXAMPLE I.

In the plane triangle ABC,

Given the sides $\begin{cases} \text{AB } 345 \text{ yards} \\ \text{AC } 232 \\ \text{BC } 174{\cdot}07 \end{cases}$



To find the angles.

1. *Geometrically.*

Draw the base AB = 345 by a scale of equal parts. With radius 232, and centre A, describe an arc; and with radius, 174, and centre B, describe another arc, cutting the former in C. Join AC, BC, and it is done.

Then, by measuring the angles, they will be found to be nearly as follows, viz.

$\angle$ A 27°, $\angle$ B $37^\circ\frac{1}{3}$, and $\angle$ C $115^\circ\frac{1}{2}$.

2. *Arithmetically.*

Having let fall the perpendicular CP, it will be,

As the base AB : AC + BC : : AC − BC : AP — BP,
that is, as 345 : 406·07 : : 57·93 : 68·18 = AP − BP,

its half is -	34·09
the half base is	172·50
the sum of these is	206·59 = AP
and their diff. is	138·41 = BP.

Then, in the triangle APC, right-angled at P,

As the side	AC - -	232	- log.	2·3654880
To sin. op.	∠ P - -	90°	- -	10·0000000
So is the side	AP - -	206·59	- -	2·3151093
To sin. op.	∠ ACP - -	62° 56′	-	9·9496213
which taken from	-	90 00		
leaves the	∠ A	27 04		

Again, in the triangle BPC, right-angled at P,

As the side	BC - -	174·07	- log.	2·2407239
To sin. op.	∠ P - -	90°	- -	10·0000000
So is side	BP - -	138·41	- -	2·1411675
To sin. op.	∠ BCP - -	52° 40′	- -	9·9004436
which taken from	-	90 00		
leaves the	∠ B	37 20		

Also the	∠ ACP	62° 56′
added to	∠ BCP	52 40
gives the whole	∠ ACB	115 36

So that all the three angles are as follow, viz.

the ∠ A 27° 4′; the ∠ B 37° 20′; the ∠ C 115° 36′.

EXAMPLE II.

In the plane triangle ABC,

Given the sides	AB 365 poles AC 154·33 BC 309·86	Ans.	∠ A 57° 12′ ∠ B 24 45 ∠ C 98 3

To find the angles.

EXAMPLE III.

In the plane triangle ABC,

Given the sides	AB 120 AC 112·65 BC 112	Ans.	∠ A 57° 27′ 0″ ∠ B 57 58 39 ∠ C 64 34 21

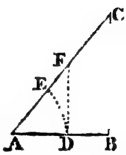
To find the angles.

The three foregoing theorems include all the cases of plane triangles, both right-angled and oblique. But there are other theorems suited to some particular forms of triangles (see vol. iii.), which are sometimes more expeditious in their use than the general ones; one of which, as the case for which it serves so frequently occurs, may be here explained.

THEOREM IV.

When a Triangle is Right-angled; any of the unknown parts may be found by the following proportions: viz.

As radius
Is to either leg of the triangle ;
So is tang. of its adjacent angle,
To its opposite leg ;
And so is secant of the same angle,
To the hypothenuse.



Demonstr. AB being the given leg, in the right-angled triangle ABC; with the centre A, and any assumed radius AD, describe an arc DE, and draw DF perpendicular to AB, or parallel to BC. Then it is evident, from the definitions, that DF is the tangent, and AF the secant of the arc DE, or of the angle A which is measured by that arc, to the radius AD. Then, because of the parallels BC, DF, it will be, - - - - as AD : AB :: DF : BC and :: AF : AC, which is the same as the theorem is in words.

Note. The radius is equal, either to the sine of 90°, or the tangent of 45°; and is expressed by 1, in a table of natural sines, or by 10 in the log. sines.

EXAMPLE 1.

In the right-angled triangle ABC,

Given $\left\{ \begin{array}{l} \text{the leg AB } 162 \\ \angle \text{A } 53^\circ\ 7'\ 48'' \end{array} \right\}$ To find AC and BC.

1. *Geometrically.*

Make AB = 162 equal parts, and the angle A = 53° 7′ 48″; then raise the perpendicular BC, meeting AC in C. So shall AC measure 270, and BC 216.

2. *Arithmetically.*

As radius	-	-	log. 10·0000000
To leg AB	-	162	2·2095150
So tang. ∠ A	-	53° 7′ 48″	10·1249371
To leg BC	-	216	2·3344521
So secant ∠ A	-	53° 7′ 48″	10·2218477
To hyp. AC	-	270	2·4313627

3. *Instrumentally.*

Extend the compasses from 45° to $53^{\circ}\frac{1}{8}$, on the tangents. Then that extent will reach from 162 to 216 on the line of numbers.

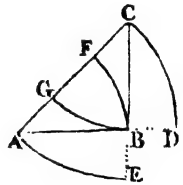
EXAMPLE II.

In the right-angled triangle ABC,

Given { the leg AB 180; the ∠ A 62° 40′ } Ans. { AC 392·0146; BC 348·2464 }

To find the other two sides.

Note. There is sometimes given another method for right-angled triangles, which is this:



ABC being such a triangle, make one leg AB radius; that is, with centre A, and distance AB, describe an arc BF. Then it is evident that the other leg BC represents the tangent, and the hypothenuse AC the secant, of the arc BF, or of the angle A.

In like manner, if the leg BC be made radius; then the other leg AB will represent the tangent, and the hypothenuse AC the secant, of the arc BG or angle C.

But if the hypothenuse be made radius; then each leg will represent the sine of its opposite angle; namely, the leg AB the sine of the arc AE or angle C, and the leg BC the sine of the arc CD or angle A.

Then the general rule for all these cases is this, namely, that the sides of the triangle bear to each other the same proportion as the parts which they represent.

And this is called, Making every side radius.

Note 2. When there are given two sides of a right-angled triangle, to find the third side; this is to be found by the property of the squares of the sides, in theorem 34, Geom. viz. that the square of the hypothenuse, or longest side, is equal to both the squares of the two other sides together.

Therefore, to find the longest side, add the squares of the two shorter sides together, and extract the square root of that sum; but to find one of the shorter sides, subtract the one square from the other, and extract the root of the remainder. Or, when the hypothenuse, H, and either the base, B, or the perpendicular, P, are given: then half the sum of log. (H + P) and log. (H − P) = log. B; and half the sum of log. (H + B) and log. (H − B) = log. P.

When B and P are given, the following logarithmic operation may sometimes be advantageously employed; viz. Find N the number answering to the log. diff., 2 log. P − log. B; and make B + N = M: then, $\frac{1}{2}$ (log. M + log. B) = log. H, the hypothenuse.

The truth of this rule is evident: for, from the nature of logarithms, $\frac{P^2}{B} = N$; whence $B + N = B + \frac{P^2}{B} = \frac{B^2 + P^2}{B} = M$; and $\frac{1}{2}$ (log. M. + log. B) = $\frac{1}{2}$ log. MB = $\frac{1}{2}$ log. $(B^2 + P^2)$ = log. $\sqrt{(B^2 + P^2)}$ = log. H.

Note, also, as many right-angled triangles in integer numbers as we please may be found by making

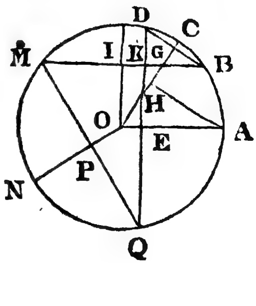
$$m^2 + n^2 = \text{hypothenuse}$$
$$m^2 - n^2 = \text{perpendicular}$$
$$2mn = \text{base}$$

m and n being taken at pleasure, m greater than n.

Before we proceed to the subject of Heights and Distances we shall give,

A CONCISE INVESTIGATION OF SOME OF THE MOST USEFUL TRIGONOMETRICAL FORMULÆ.

Let AB, AC, AD, be three arches, such that BC = CD, and O the centre. Join AO, OC, BD. Draw DEQ and OI perpendicular, and BIM ∥ to OA. Join MQ and bisect it by the radius ON; and draw AH ∥ to BD.



Then is AH = sin. AC

OH = cos. AC;

also DE = EQ = sin. AD

EK = OI = sin. AB

QK = sin. AD + sin. AB

DK = sin. AD − sin. AB

BI = IM = cos. AB

OE = KI = cos. AD

MK = cos. AB + cos. AD

BK = cos. AB − cos. AD

Because the angles at K are right angles:

$$\text{arc BD} + \text{arc MQ} = 180^\circ, \text{ and arc DC} + \text{arc MN} = 90^\circ$$
$$\therefore \text{MP} = \text{PQ} = \text{OG} = \cos. \text{DC} = \cos. \text{BC};$$

also, because $\text{AC} = \frac{1}{2}(\text{AB} + \text{AD}) = \frac{1}{2}\text{BAQ} = \text{angle AOC}$ (at centre) $= \text{BDQ}$ (at circumf.) $= \text{BMQ}$ (on same arc)

∴ triangles AOH, BDK, QMK, are equiangular.

Hence—

$$\text{I. OA} : \text{AH} :: \text{MQ} : \text{QK};$$
that is, $\text{rad.} : \sin. \text{AC} :: 2\cos. \text{BC} : \sin. \text{AD} + \sin. \text{AB}$
$$\text{II. AD} : \text{DH} :: \text{BD} : \text{DK};$$
or, $\text{rad.} : \cos. \text{AC} :: 2\sin. \text{BC} : \sin. \text{AD} - \sin. \text{AB}$
$$\text{III. AO} : \text{OH} :: \text{QM} : \text{MK};$$
or, $\text{rad.} : \cos. \text{AC} :: 2\cos. \text{BC} : \cos. \text{AB} + \cos. \text{AD}$
$$\text{IV. AO} : \text{AH} :: \text{DB} : \text{BK};$$
or, $\text{rad.} : \sin. \text{AC} :: 2\sin. \text{BC} : \cos. \text{AB} - \cos. \text{AD};$

also,

V. $\text{BK} \cdot \text{KM} = \text{DK} \cdot \text{KQ}$, that is $(\cos. \text{AB} - \cos. \text{AD})(\cos. \text{AB} + \cos. \text{AD}) = (\sin. \text{AD} - \sin. \text{AB})(\sin. \text{AD} + \sin. \text{AB})$.

By reducing the above four proportions into equations, making rad. = 1, we obtain two distinct classes of formulæ, thus:—

First Class. $\text{AC} = a$, $\text{CB} = b$; then $\text{AD} = a + b$, $\text{AB} = a - b$,

1. $\sin. (a + b) + \sin. (a - b) = 2 \sin. a \cos. b$
2. $\sin. (a + b) - \sin. (a - b) = 2 \cos. a \sin. b$
3. $\cos. (a - b) + \cos. (a + b) = 2 \cos. a \cos. b$
4. $\cos. (a - b) - \cos. (a + b) = 2 \sin. a \sin. b$

Second Class. $\text{AD} = a$, $\text{AB} = b$; then $\text{AC} = \frac{1}{2}(a + b)$, $\text{BC} = \frac{1}{2}(a - b)$.

5. $\sin. a + \sin. b = 2 \sin. \frac{1}{2}(a + b) \cos. \frac{1}{2}(a - b)$
6. $\sin. a - \sin. b = 2 \cos. \frac{1}{2}(a + b) \sin. \frac{1}{2}(a - b)$
7. $\cos. b + \cos. a = 2 \cos. \frac{1}{2}(a + b) \cos. \frac{1}{2}(a - b)$
8. $\cos. b - \cos. a = 2 \sin. \frac{1}{2}(a + b) \sin. \frac{1}{2}(a - b)$

The first class is useful in transforming the products of sines into simple sines, and the contrary.

The second facilitates the substitution of sums or differences of sines for the products, and the contrary.

Taking the sum and the difference of equations 1 and 2, also of 3 and 4, remembering that sin. = cos. tan. we obtain the following:

Third Class.

$$9.\quad \sin.(a+b) = \sin. a \cos. b + \sin. b \cos. a = \cos. a \cos. b\,(\tan. a + \tan. b)$$

$$10.\quad \sin.(a-b) = \sin. a \cos. b - \sin. b \cos. a = \cos. a \cos. b\,(\tan. a - \tan. b)$$

$$11.\quad \cos.(a+b) = \cos. a \cos. b - \sin. a \sin. b = \cos. a \cos. b\,(1 - \tan. a \tan. b)$$

$$12.\quad \cos.(a-b) = \cos. a \cos. b + \sin. a \sin. b = \cos. a \cos. b\,(1 + \tan. a \tan. b).$$

From these, making $a = b$, we readily obtain the expressions for sines and cosines of double arcs; also dividing equation 9 by 11, and equation 10 by 12, we obtain expressions for the tangents of $a + b$ and $a - b$. Thus we have:—

Fourth Class.

$$13.\quad \sin. 2a = 2 \sin. a \cos. a = 2 \cos.^2 a \tan. a$$

$$14.\quad \cos. 2a = \cos.^2 a - \sin.^2 a = \cos.^2 a\,(1 - \tan.^2 a)$$

$$15.\quad \frac{\sin.}{\cos.}(a + b) = \tan.(a + b) = \frac{\tan. a + \tan. b}{1 - \tan. a \tan. b}$$

$$16.\quad \frac{\sin.}{\cos.}(a - b) = \tan.(a - b) = \frac{\tan. a - \tan. b}{1 + \tan. a \tan. b}$$

$$17.\quad \tan. 2a = \frac{2 \tan. a}{1 - \tan.^2 a}$$

$$18.\quad \cot. 2a = \frac{1 - \tan.^2 a}{2 \tan. a}.$$

Substituting in the second class,
for $\sin. \frac{1}{2}(a + b)$, $\cos. \frac{1}{2}(a + b) \tan. \frac{1}{2}(a + b)$,
and for $\sin. \frac{1}{2}(a - b)$, $\cos. \frac{1}{2}(a - b) \tan. \frac{1}{2}(a - b)$, we have:—

Fifth Class.

$$19.\quad \cos. b + \cos. a = 2 \cos. \tfrac{1}{2}(a+b) \cos. \tfrac{1}{2}(a-b).\text{—See equa. 7.}$$

$$20.\quad \cos. b - \cos. a = \tan. \tfrac{1}{2}(a+b) \tan. \tfrac{1}{2}(a-b)\, 2 \cos. \tfrac{1}{2}(a+b) \cos. \tfrac{1}{2}(a-b) = \tan. \tfrac{1}{2}(a+b) \tan. \tfrac{1}{2}(a-b)\,(\cos. b + \cos. a)$$

$$21.\quad \sin. a + \sin. b = \tan. \tfrac{1}{2}(a+b)\, 2 \cos. \tfrac{1}{2}(a+b) \cos. \tfrac{1}{2}(a-b) = \tan. \tfrac{1}{2}(a+b)\,(\cos. a + \cos. b)$$

$$22.\quad \sin. a - \sin. b = \tan. \tfrac{1}{2}(a-b)\, 2 \cos. \tfrac{1}{2}(a+b) \cos. \tfrac{1}{2}(a-b) = \tan. \tfrac{1}{2}(a-b)\,(\cos. a + \cos. b)$$

$$23.\quad \frac{\sin. a + \sin. b}{\sin. a - \sin. b} = \frac{\tan. \frac{1}{2}(a+b)}{\tan. \frac{1}{2}(a-b)}\text{: from 21 and 22.}$$

24. $\dfrac{\sin. a + \sin. b}{\cos. a + \cos. b} = \tan. \tfrac{1}{2}(a + b)$: from 21.

25. $\dfrac{\sin. a - \sin. b}{\cos. a + \cos. b} = \tan. \tfrac{1}{2}(a - b)$: from 22.

Examples for Exercise.

1. Demonstrate that in any right-angled plane triangle the following properties obtain: viz.

(1.) $\dfrac{\text{perp.}}{\text{base}} = \text{tan. ang. at base.}$ (2.) $\dfrac{\text{base}}{\text{perp.}} = \text{tan. ang. at vertex.}$

(3.) $\dfrac{\text{perp.}}{\text{hyp.}} = \text{sin. ang. at base.}$ (4.) $\dfrac{\text{base}}{\text{hyp.}} = \text{sin. ang. at vertex.}$

(5.) $\dfrac{\text{hyp.}}{\text{base}} = \text{sec. ang. at base.}$ (6.) $\dfrac{\text{hyp.}}{\text{perp.}} = \text{sec. ang. at vertex.}$

2. Demonstrate that sec. $A = \tan. (45° + \tfrac{1}{2}A)$.

3. Demonstrate that sec. $2A = \dfrac{1 + \tan.^2 A}{1 - \tan.^2 A}$, and that

cosec. $2A = \dfrac{1 + \tan.^2 A}{2 \tan. A} = \dfrac{\sec.^2 A}{2 \tan. A}$.

4. Given $Axy = By^2 + Dx^2$; to find x and y the sine and cosine of an arc.

5. Demonstrate that of any arc, $\tan.^2 - \sin.^2 = \tan.^2 \sin.^2$.

6. Demonstrate that if the tan. of an arc be $= \sqrt{n}$, the sine of the same arc is $= \sqrt{\dfrac{n}{n+1}}$.

OF HEIGHTS AND DISTANCES, &c.

By the mensuration and protraction of lines and angles, are determined the lengths, heights, depths, and distances of bodies or objects.

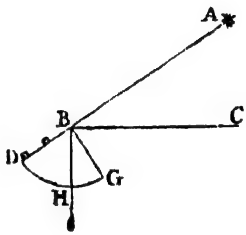
Accessible lines are measured by applying to them some certain measure a number of times, as an inch, or a foot, or

yard. But inaccessible lines must be measured by taking angles, or by such-like method, drawn from the principles of geometry.

When instruments are used for taking the magnitude of the angles in degrees, the lines are then calculated by trigonometry: in the other methods, the lines are calculated from the principle of similar triangles, or some other geometrical property, without regard to the measure of the angles.

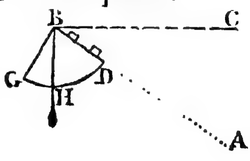
Angles of elevation, or of depression, are usually taken either with a theodolite, or with a quadrant, divided into degrees and minutes, and furnished with a plummet suspended from the centre, and two open sights fixed on one of the radii, or else with telescopic sights.

To take an Angle of Altitude and Depression with the Quadrant.



Let A be any object, as the sun, moon, or a star, or the top of a tower, or hill, or other eminence: and let it be required to find the measure of the angle ABC, which a line drawn from the object makes above the horizontal line BC.

Place the centre of the quadrant in the angular point, and move it round there as a centre, till with one eye at D, the other being shut, you perceive the object A through the sights; then will the arc GH of the quadrant, cut off by the plumb-line BH, be the measure of the angle ABC as required.



The angle ABC of depression of any object A, below the horizontal line BC, is taken in the same manner; except that here the eye is applied to the centre, and the measure of the angle is the arc GH, on the other side of the plumb-line.

The following examples are to be constructed and calculated by the rules of Trigonometry.

EXAMPLE I.

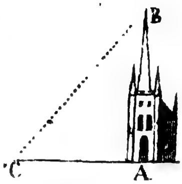
Having measured a distance of 200 feet, in a direct horizontal line, from the bottom of a steeple, the angle of elevation of its top, taken at that distance, was found to be 47° 30′; hence it is required to find the height of the steeple.

Construction.

Draw an indefinite line; on which set off AC = 200 equal parts, for the measured distance. Erect the indefinite perpendicular AB; and draw CB so as to make the angle C = 47° 30′, the angle of elevation; and it is done. Then AB, measured on the scale of equal parts, is nearly

Calculation.

As radius	- -	10·0000000
To AC 200	- -	2·3010300
So tang. ∠ C 47° 30′		10·0379475
To AB 218·26 required		2·3389775



Or, by the nat. tangents, we have AC × tan. BCA = 200 × 1·091308 = 218·2616 = AB.

EXAMPLE II.

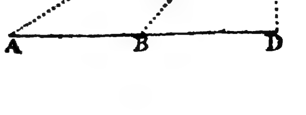
What was the perpendicular height of a cloud, or of a balloon, when its angles of elevation were 35° and 64°, as taken by two observers, at the same time, both on the same side of it, and in the same vertical plane; the distance between them being half a mile or 880 yards? And what was its distance from the said two observers?

Construction.

Draw an indefinite ground line, on which set off the given distance AB = 880; then A and B are the places of the observers. Make the angle A = 35°, and the angle B = 64°; then the intersection of the lines at C will be the place of the balloon: whence the perpendicular CD, being let fall, will be its perpendicular height. Then by measurement are found the distances and height nearly as follow, viz. AC 1631, BC 1041, DC 936.

Calculation.

First, from ∠ B		
take	∠ A	35
leaves	∠ ACB	29



Then in the triangle ABC,

As sin. ∠ ACB	29°	- - -	9·6855712
To op. side AB	880	- - -	2·9444827
So sin. ∠ A	35°	- - -	9·7585913
To op. side BC	1041·125	- -	3·0175028

As sin. ∠ ACB	29°	- - -	9·6855712
To op. side AB	880	- - -	2·9444827
So sin. ∠ B 116° or 64°		- - -	9·9536602
To op. side AC	1631·442	- - -	3·2125717

And in the triangle BCD,

As sin. ∠ D	90°	- - -	10·0000000
To op. side BC	1041·125	- -	3·0175028
So sin. ∠ B	64°	- - -	9·9536602
To op. side CD	935·757	- -	2·9711630

EXAMPLE III.

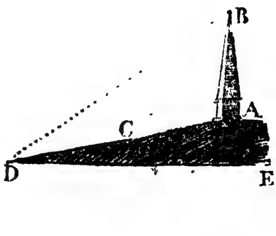
Having to find the height of an obelisk standing on the top of a declivity, I first measured from its bottom a distance of 40 feet, and there found the angle, formed by the oblique plane and a line imagined to go to the top of the obelisk, 41°; but after measuring on in the same direction 60 feet farther, the like angle was only 23° 45′. What then was the height of the obelisk?

Construction.

Draw an indefinite line for the sloping plane or declivity, in which assume any point A for the bottom of the obelisk, from which set off the distance AC = 40, and again CD = 60 equal parts. Then make the angle C = 41°, and the angle D = 23° 45′; and the point B where the two lines meet will be the top of the obelisk. Therefore AB, joined, will be its height.—Draw also the horizontal line DE perp. to AB.

Calculation.

From the ∠ C	41°	00′
take the ∠ D	23	45
leaves the ∠ DBC	17	15



Then in the triangle DBC,

As sin. ∠DBC	17° 15′	- - -	9·4720856
To op. side DC	60	- - -	1·7781513
So sin. ∠D	23 45	- - -	9·6050320
To op. side CB	81·488	- - -	1·9110977

And in the triangle ABC,

As sum of sides CB, CA	121·488	-	2·0845333
To diff. of sides CB, CA	41·488	-	1·6179225
So tang. $\frac{1}{2}$(A + B)	69° 30′	-	10·4272623
To tang. $\frac{1}{2}$(A − B)	42 24$\frac{1}{2}$	-	9·9606516

the diff. of these is ∠CBA 27 5$\frac{1}{2}$
the sum is ∠CAB 111 54$\frac{1}{2}$

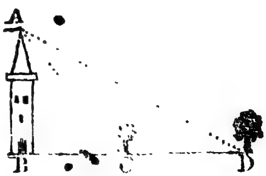
Lastly, as sin. ∠CBA	27° 5′$\frac{1}{2}$	- -	9·6582842
To op. side CA	40	- -	1·6020600
So sin. ∠C	41° 0′	- -	9·8169429
To op. side AB	57·623	- -	1·7607187

Also the ∠ADB = BAC − 90° = 21° 54′$\frac{1}{2}$

EXAMPLE IV.

Wanting to know the distance between two inaccessible trees, or other objects, from the top of a tower 120 feet high, which lay in the same right line with the two objects, I took the angles formed by the perpendicular wall and lines conceived to be drawn from the top of the tower to the bottom of each tree, and found them to be 33° and 64°$\frac{1}{2}$. What is the distance between the two objects?

Construction.



Draw the indefinite ground line BD, and perpendicular to it BA = 120 equal parts. Then draw the two lines AC, AD, making the two angles BAC, BAD, equal to the given measures 33° and 64°$\frac{1}{2}$. So shall C and D be the places of the two objects.

Calculation.

First, in the right-angled triangle ABC,

As radius	- - - - -		10·0000000
To AB	-	120 - - -	2·0791812
So tang. ∠BAC		33° - - -	9·8125174
To BC	-	77·929 - - -	1·8916986

Then in the right-angled triangle ABD,

As radius	- - - - -		10·0000000
To AB	- -	120 - -	2·0791812
So tang. ∠BAD	-	64° 30′ -	10·3215039
To BD	-	251·585 - -	2·4006851

From which take BC 77·929
leaves the dist. CD 173 656, as required.

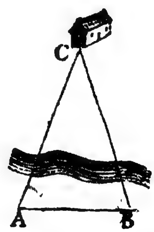
Or thus, by the natural tangents,

From nat. tan. DAB	- -	64° 30′	= 2·0965436
Take nat. tan. CAB	- -	33 0	= 0·6494076
Difference	- - -		1·4471360
If drawn into AB	- -		120
The result gives CD	- -		= 173·65632

EXAMPLE V.

Being on the side of a river, and wanting to know the distance to a house which was seen on the other side, I measured 200 yards in a straight line by the side of the river; and then, at each end of this line of distance, took the horizontal angle formed between the house and the other end of the line; which angles were, the one of them 68° 2′, and the other 73° 15′. What were the distances from each end to the house?

Construction.



Draw the line AB = 200 equal parts. Then draw AC so as to make the angle A = 68° 2′, and BC to make the angle B = 73° 15′. So shall the point C be the place of the house required.

The calculation, which is left for the student's exercise, gives AC = 306·19, BC = 296·54.

EXAM. VI. From the edge of a ditch, of 36 feet wide, surrounding a fort, having taken the angle of elevation of the top of the wall, it was found to be 62° 40′: required the height of the wall, and the length of a ladder to reach from my station to the top of it?

Ans. $\begin{cases} \text{height of wall } 69{\cdot}64, \\ \text{ladder, } 78{\cdot}4 \text{ feet.} \end{cases}$

EXAM. VII. Required the length of a shoar, which being to strut 11 feet from the upright of a building, will support a jamb 23 feet 10 inches from the ground?

Ans. 26 feet 3 inches.

EXAM. VIII. A ladder, 40 feet long, can be so placed, that it shall reach a window 33 feet from the ground, on one side of the street; and by turning it over, without moving the foot out of its place, it will do the same by a window 21 feet high, on the other side: required the breadth of the street?

Ans. 56·649 feet.

EXAM. IX. A maypole, whose top was broken off by a blast of wind, struck the ground at 15 feet distance from the foot of the pole: what was the height of the whole maypole, supposing the broken piece to measure 39 feet in length?

Ans. 75 feet.

EXAM. X. At 170 feet distance from the bottom of a tower, the angle of its elevation was found to be 52° 30′: required the altitude of the tower? Ans. 221·55 feet.

EXAM. XI. From the top of a tower, by the sea-side, of 143 feet high, it was observed that the angle of depression of a ship's bottom, then at anchor, measured 35°; what was the ship's distance from the bottom of the wall?

Ans. 204·22 feet.

EXAM. XII. What is the perpendicular height of a hill; its angle of elevation, taken at the bottom of it, being 46°, and 200 yards farther off, on a level with the bottom, the angle was 31°? Ans. 286·28 yards.

EXAM. XIII. Wanting to know the height of an inaccessible tower; at the least distance from it, on the same horizontal plane, I took its angle of elevation equal to 58°; then going 300 feet directly from it, found the angle there to be only 32°: required its height, and my distance from it at the first station?

Ans. $\begin{cases} \text{height } 307{\cdot}53 \\ \text{distance } 192{\cdot}15 \end{cases}$

EXAM. XIV. Being on a horizontal plane, and wanting to know the height of a tower placed on the top of an inaccessible hill; I took the angle of elevation of the top of the hill

40°, and of the top of the tower 51°; then measuring in a line directly from it to the distance of 200 feet farther, I found the angle to the top of the tower to be 33° 45′. What is the height of the tower? Ans. 93·33148 feet.

Exam. xv. From a window near the bottom of a house, which seemed to be on a level with the bottom of a steeple, I took the angle of elevation of the top of the steeple equal 40°; then from another window, 18 feet directly above the former, the like angle was 37° 30′: required the height and distance of the steeple.

Ans. { height 210·44
distance 250·79 }

Exam. xvi. Wanting to know the height of, and my distance from, an object on the other side of a river, which appeared to be on a level with the place where I stood, close by the side of the river; and not having room to measure backward, in the same line, because of the immediate rise of the bank, I placed a mark where I stood, and measured in a direction from the object, up the ascending ground, to the distance of 264 feet, where it was evident that I was above the level of the top of the object; there the angles of depression were found to be, viz. of the mark left at the river's side 42°, of the bottom of the object 27°, and of its top 19°. Required the height of the object, and the distance of the mark from its bottom?

Ans. { height 57·26
distance 150·50 }

Exam. xvii. If the height of the mountain called the Peak of Teneriffe be $2\frac{1}{3}$ miles, as it is very nearly, and the angle taken at the top of it, as formed between a plumb-line and a line conceived to touch the earth in the horizon, or farthest visible point, be 88° 2′; it is required from these measures to determine the magnitude of the whole earth, and the utmost distance that can be seen on its surface from the top of the mountain, supposing the form of the earth to be perfectly globular?

Ans. { dist. 135·943
diam. 7916 } miles.

Exam. xviii. Two ships of war, intending to cannonade a fort, are, by the shallowness of the water, kept so far from it, that they suspect their guns cannot reach it with effect. In order therefore to measure the distance, they separate from each other a quarter of a mile, or 440 yards; then each ship observes and measures the angle which the other ship

and the fort subtends, which angles are 83° 45′ and 85° 15′. What is the distance between each ship and the fort?

Ans. $\begin{cases} 2292 \cdot 26 \text{ yards.} \\ 2298 \cdot 05 \end{cases}$

Exam. xix. Wanting to know the breadth of a river, I measured a base of 500 yards in a straight line close by one side of it; and at each end of this line I found the angles subtended by the other end and a tree, close to the bank on the other side of the river, to be 53° and 79° 12′. What was the perpendicular breadth of the river?

Ans. 529·48 yards.

Exam. xx. Wanting to know the extent of a piece of water, or distance between two headlands; I measured from each of them to a certain point inland, and found the two distances to be 735 yards and 840 yards; also the horizontal angle subtended between these two lines was 55° 40′. What was the distance required? Ans. 741·2 yards.

Exam. xxi. A point of land was observed, by a ship at sea, to bear east-by-south; and after sailing north-east 12 miles, it was found to bear south-east-by-east. It is required to determine the place of that headland, and the ship's distance from it at the last observation? Ans. 26·0728 miles.

Exam. xxii. Wanting to know the distance between a house and a mill, which were seen at a distance on the other side of a river, I measured a base line along the side where I was, of 600 yards, and at each end of it took the angles subtended by the other end and the house and mill, which were as follow, viz. at one end the angles were 58° 20′ and 95° 20′, and at the other end the like angles were 53° 30′ and 98° 45′. What then was the distance between the house and mill? Ans. 962·5866 yards.

Exam. xxiii. Wanting to know my distance from an inaccessible object o, on the other side of a river; and having no instrument for taking angles, but only a chain or cord for measuring distances; from each of two stations, A and B, which were taken at 500 yards asunder, I measured in a direct line from the object o 100 yards, viz. AC and BD each equal to 100 yards; also the diagonal AD measured 550 yards, and the diagonal BC 560. What was the distance of the object o from each station A and B?

Ans. $\begin{cases} \text{AO } 536 \cdot 81 \\ \text{BO } 500 \cdot 47 \end{cases}$

Exam. xxiv. In a garrison besieged are three remarkable objects, A, B, C, the distances of which from each other are

discovered by means of a map of the place, and are as follow, viz. AB $266\frac{1}{4}$, AC 530, BC $327\frac{1}{2}$ yards. Now, having to erect a battery against it, at a certain spot without the place, and being desirous to know whether my distances from the three objects be such, as that they may from thence be battered with effect, I took, with an instrument, the horizontal angles subtended by these objects from the station S, and found them to be as follow, viz. the angle ASB 13° 30′, and the angle BSC 29°.50′. Required the three distances, SA, SB, SC; the object B being situated nearest to me, and between the two others A and C.

Ans. $\begin{cases} \text{SA } 757\cdot14 \\ \text{SB } 537\cdot10 \\ \text{SC } 655\cdot30 \end{cases}$

EXAM. XXV. Required the same as in the last example, when the object B is the farthest from my station, but still seen between the two others as to angular position, and those angles being thus, the angle ASB 33° 45′, and BSC 22° 30′, also the three distances, AB 600, AC 800, BC 400 yards?

Ans. $\begin{cases} \text{SA } 710\cdot3 \\ \text{SB } 1041\cdot85 \\ \text{SC } 934\cdot14 \end{cases}$

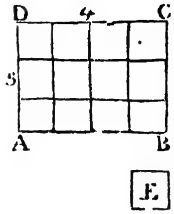
EXAM. XXVI. If DB in the figure at pa. 2 represent a portion of the earth's surface, and D the point where the levelling instrument is placed, then LB will be the difference between the true and the apparent level; and you are required to demonstrate that, for distances not exceeding 5 or 6 miles measured on the earth's surface, BL, estimated in feet, is equal to $\frac{2}{3}$ of the square of BD, taken in miles.

MENSURATION OF PLANES.

THE Area of any plane figure, is the measure of the space contained within its extremes or bounds; without any regard to thickness.

This area, or the content of the plane figure, is estimated by the number of little squares that may be contained in it; the side of those little measuring squares being an inch, or a foot, or a yard, or any other fixed quantity. And hence the area or content is said to be so many square inches, or square feet, or square yards, &c.

Thus, if the figure to be measured be the rectangle ABCD, and the little square E, whose side is one inch, be the measuring unit proposed: then as often as the said little square is contained in the rectangle, so many square inches the rectangle is said to contain, which in the present case is 12.



PROBLEM I.

To find the Area of any Parallelogram; whether it be a Square, a Rectangle, a Rhombus, or a Rhomboid.

MULTIPLY the length by the perpendicular breadth, or height, and the product will be the area *.

EXAMPLES.

Ex. 1. To find the area of a parallelogram, the length being 12·25, and breadth or height 8·5.

$$\begin{array}{rl} 12{\cdot}25 & \text{length} \\ 8{\cdot}5 & \text{breadth} \\ \hline 6125 & \\ 9800 & \\ \hline 104{\cdot}125 & \text{area.} \end{array}$$

* The truth of this rule is proved in the Geom. theor. 81, cor. 2.

The same is otherwise proved thus: Let the foregoing rectangle be the figure proposed; and let the length and breadth be divided into several parts, each equal to the linear measuring unit, being here 4 for the length, and 3 for the breadth; and let the opposite points of division be connected by right lines.—Then it is evident that these lines divide the rectangle into a number of little squares, each equal to the square measuring unit E; and further, that the number of these little squares, or the area of the figure, is equal to the number of linear measuring units in the length, repeated as often as there are linear measuring units in the breadth, or height; that is, equal to the length drawn into the height; which here is 4 × 3 or 12.

And it is proved (Geom. theor. 25, cor. 2), that any oblique parallelogram is equal to a rectangle, of equal length and perpendicular breadth. Therefore the rule is general for all parallelograms whatever.

Ex. 2. To find the area of a square, whose side is 35·25 chains. Ans. 124 acres, 1 rood, 1 perch.

Ex. 3. To find the area of a rectangular board, whose length is $12\frac{1}{2}$ feet, and breadth 9 inches. Ans. $9\frac{3}{8}$ feet.

Ex. 4. To find the content of a piece of land, in form of a rhombus, its length being 6·20 chains, and perpendicular breadth 5·45. Ans. 3 acres, 1 rood, 20 perches.

Ex. 5. To find the number of square yards of painting in a rhomboid, whose length is 37 feet, and height 5 feet 3 inches. Ans. $21\frac{7}{12}$ square yards.

PROBLEM II.

To find the Area of a Triangle.

Rule 1. Multiply the base by the perpendicular height, and take half the product for the area *. Or, multiply the one of these dimensions by half the other.

EXAMPLES.

Ex. 1. To find the area of a triangle, whose base is 625, and perpendicular height 520 links?

Here $625 \times 260 = 1{\cdot}62500$ square links,

or equal 1 acre, 2 roods, 20 perches, the answer.

Ex. 2. How many square yards contains the triangle, whose base is 40, and perpendicular 30 feet?

Ans. $66\frac{2}{3}$ square yards.

Ex. 3. To find the number of square yards in a triangle, whose base is 49 feet, and height $25\frac{1}{4}$ feet?

Ans. $68\frac{53}{72}$, or 68·7361.

Ex. 4. To find the area of a triangle, whose base is 18 feet 4 inches, and height 11 feet 10 inches?

Ans. 108 feet, $5\frac{2}{3}$ inches.

Rule II. When two sides and their contained angle are given: Multiply the two given sides together, and take half their product: Then say, as radius is to the sine of the given angle, so is that half product, to the area of the triangle.

* The truth of this rule is evident, because any triangle is the half of a parallelogram of equal base and altitude, by Geom. theor. 26.

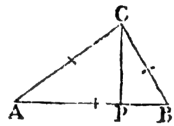
Or, multiply that half product by the natural sine of the said angle, for the area *.

Ex. 1. What is the area of a triangle, whose two sides are 30 and 40, and their contained angle 28° 57′?

By Natural Numbers.	*By Logarithms.*
First, $\frac{1}{2} \times 40 \times 30 = 600$,	
then, 1 : 600 : : ·484046 sin. 28° 57′	log. 9·684887
600	2·778151
Answer 290·4276 the area answ. to	2·463038

Ex. 2. How many square yards contains the triangle of which one angle is 45°, and its containing sides 25 and $21\frac{1}{4}$ feet? Ans. 20·86947.

Rule III. When the three sides are given: Add all the three sides together, and take half that sum. Next, subtract each side severally from the said half sum, obtaining three remainders. Then multiply the said half sum and those three remainders all together, and extract the square root of the last product, for the area of the triangle †.



* For, let AB, AC, be the two given sides, including the given angle A. Now $\frac{1}{2}$AB × CP is the area, by the first rule, CP being the perpendicular. But by trigonometry, as sin. ∠P, or radius : AC : : sin. ∠A : CP, which is therefore = AC × sin. ∠A, taking radius = 1. Therefore the area $\frac{1}{2}$AB × CP is = $\frac{1}{2}$AB × AC × sin. ∠A, to radius 1; or, as radius : sin. ∠A : : $\frac{1}{2}$AB × AC : the area.

† For, let b denote the base AB of the triangle ABC (see the last fig.), also a the side AC, and c the side BC. Then, by theor. 3, Trigon. as $b : a + c : : a - c : \frac{aa - cc}{b}$ = AP − PB the diff. of the segments;

theref. $\frac{1}{2}b + \frac{aa - cc}{2b} = \frac{bb + aa - cc}{2b}$ = the segment AP;

hence $\sqrt{AC^2 - AP^2}$ = the perp. CP, that is,

$$\sqrt{\left(aa - \left(\frac{bb + aa - cc}{2b}\right)^2\right)} = \text{- - - - - - -}$$

$$\sqrt{\frac{2a^2b^2 - a^4 + 2b^2c^2 - b^4 + 2a^2c^2 - c^4}{4bb}} = \text{CP}.$$

But $\frac{1}{2}$AB × CP is the area, that is,

If the sides of the triangle be large, then add the logs. of the half sum, and of the three remainders together, and half their sum will be the log. of the area.

Ex. 1. To find the area of the triangle whose three sides are 20, 30, 40.

20	45	45	45
30	20	30	40
40	—	—	—
——	25 1st rem.	15 2d rem.	5 3d rem.
2) 90	—	—	—
45 half sum			
—			

Then $45 \times 25 \times 15 \times 5 = 84375$,
The root of which is 290·4737, the area.

Ex. 2. How many square yards of plastering are in a triangle, whose sides are 30, 40, 50 feet? Ans. $66\frac{2}{3}$.

Ex. 3. How many acres, &c. contains the triangle, whose sides are 2569, 4900, 5025 links?

Ans. 61 acres, 1 rood, 39 perches.

PROBLEM III.

To find the Area of a Trapezoid.

ADD together the two parallel sides; then multiply their sum by the perpendicular breadth, or the distance between them; and take half the product for the area. By Geom. theor. 29.

$$\tfrac{1}{2}b \times \text{CP} = \sqrt{\frac{2a^2b^2-a^4+2b^2c^2-b^4+2a^2c^2-c^4}{16}}$$

$$= \sqrt{\left(\frac{aa-bb-cc+2bc}{4} \times \frac{-aa+bb+cc+2bc}{4}\right)} \quad \text{(A)}$$

$$= \sqrt{\left(\frac{a+b+c}{2} \times \frac{-a+b+c}{2} \times \frac{a-b+c}{2} \times \frac{a+b-c}{2}\right)}$$

$= \sqrt{\{s \times (s-a) \times (s-b) \times (s-c)\}}$, which is the rule, where s denotes half the sum of the three sides.

The expression marked (A), if we put $s = b + c$, and d for $b - c$, is equivalent to $\frac{1}{4}\sqrt{\{(a^2-d^2)(s^2-d^2)\}}$; which, in most cases, furnishes a more commodious rule for practice than rule III. here given; especially if the computer have a table of squares at hand.

Ex. 1. In a trapezoid, the parallel sides are 750 and 1225, and the perpendicular distance between them 1540 links: to find the area.

$$\begin{array}{r} 1225 \\ 750 \end{array}$$

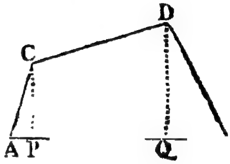
$1975 \times 770 = 152075$ square links $= 15$ acr. 33 perc.

Ex. 2. How many square feet are contained in the plank, whose length is 12 feet 6 inches, the breadth at the greater end 15 inches, and at the less end 11 inches?

Ans. $13\frac{13}{24}$ feet.

Ex. 3. In measuring along one side AB of a quadrangular field, that side, and the two perpendiculars let fall on it from the two opposite corners, measured as follow: required the content.

AP = 110 links
AQ = 745
AB = 1110
CP = 352
DQ = 595



Ans. 4 acres, 1 rood, 5·792 perches.

PROBLEM IV.

To find the Area of any Trapezium.

Divide the trapezium into two triangles by a diagonal; then find the areas of these triangles, and add them together.

Or thus, let fall two perpendiculars on the diagonal from the other two opposite angles; then add these two perpendiculars together, and multiply that sum by the diagonal, taking half the product for the area of the trapezium.

Ex. 1. To find the area of the trapezium, whose diagonal is 42, and the two perpendiculars on it 16 and 18.

Here $16 + 18 = 34$, its half is 17.
Then $42 \times 17 = 714$ the area.

Ex. 2. How many square yards of paving are in the trapezium, whose diagonal is 65 feet, and the two perpendiculars let fall on it 28 and $33\frac{1}{2}$ feet? Ans. $222\frac{1}{12}$ yards.

Ex. 3. In the quadrangular field ABCD, on account of obstructions there could only be taken the following measures, viz. the two sides BC 265 and AD 220 yards; the diagonal AC 378, and the two distances of the perpendiculars from the

ends of the diagonal, namely, AE 100, and CF 70 yards. Required the construction of the figure, and the area in acres, when 4840 square yards make an acre?

Ans. 17 acres, 2 roods, 21 perches.

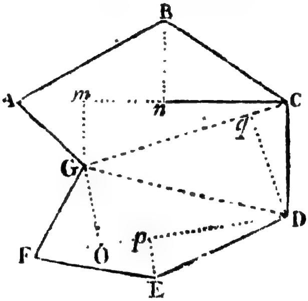
PROBLEM V.

To find the Area of an Irregular Polygon.

DRAW diagonals dividing the proposed polygon into trapeziums and triangles. Then find the areas of all these separately, and add them together for the content of the whole polygon.

EXAMPLE. To find the content of the irregular figure ABCDEFGA, in which are given the following diagonals and perpendiculars: namely,

AC 55
FD 52
GC 44
Gm 13
Bn 18
GO 12
Ep 8
Dq 23



Ans. 1878½.

PROBLEM VI.

To find the Area of a Regular Polygon.

RULE I. MULTIPLY the perimeter of the polygon, or sum of its sides, by the perpendicular drawn from its centre on one of its sides, and take half the product for the area*.

Ex. 1. To find the area of a regular pentagon, each side being 25 feet, and the perpendicular from the centre on each side 17·2047737.

Here $25 \times 5 = 125$ is the perimeter.
And $17{\cdot}2047737 \times 125 = 2150{\cdot}5967125$.
Its half 1075·298356 is the area sought.

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* This is only in effect resolving the polygon into as many equal triangles as it has sides, by drawing lines from the centre to all the angles; then finding their areas, and adding them all together.

RULE II. Square the side of the polygon; then multiply that square by the tabular area, or multiplier set against its name in the following table, and the product will be the area*.

No. of Sides.	Names.	Areas, or Multipliers.	Radius of circum. circle.
3	Trigon or triangle	0·4330127	0·5773503
4	Tetragon or square	1·0000000	0·7071068
5	Pentagon	1·7204774	0·8506508
6	Hexagon	2·5980762	1·0000000
7	Heptagon	3·6339124	1·1523824
8	Octagon	4·8284271	1·3065628
9	Nonagon	6·1818242	1·4619022
10	Decagon	7·6942088	1·6180340
11	Undecagon	9·3656399	1·7747324
12	Dodecagon	11·1961524	1·9318517

EXAM. Taking here the same example as before, namely, a pentagon, whose side is 25 feet.

Then 25^2 being $= 625$,
And the tabular area 1·7204774;
Theref. $1{\cdot}7204774 \times 625 = 1075{\cdot}298375$, as before.

Ex. 2. To find the area of the trigon or equilateral triangle, whose side is 20. Ans. 173·20508.

Ex. 3. To find the area of the hexagon whose side is 20. Ans. 1039·23048.

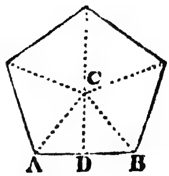
Ex. 4. To find the area of an octagon whose side is 20. Ans. 1931·37084.

Ex. 5. To find the area of a decagon whose side is 20. Ans. 3077·68352.

Note. If AB $= 1$, and n the number of sides of the polygon, then area of polygon $= n$ times area of the triangle ABC $= n$ AD . DC $= n$ AD tan. CAD (to rad AD) $= \frac{1}{4}n$ tan. CAD

* This rule is founded on the property, that like polygons, being similar figures, are to one another as the squares of their like sides; which is proved in the Geom. theor. 89. Now, the multipliers in the table, are the areas of the respective polygons to the side 1. Whence the rule is manifest.

$= \frac{1}{4}n$ cot. ACD $= \frac{1}{4}n$ cot. $\frac{180^\circ}{n}$. The radius of the circumscribing circle, to side 1, is evidently equal to $\frac{1}{2}$ sec. CAD. Multiplying, therefore, the radius of the table by the numeral value of any proposed side, the product is the radius of a circle in which that polygon may be inscribed; and from which it may readily be constructed.



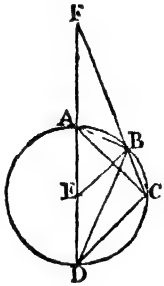
PROBLEM VII.

To find the Diameter and Circumference of any Circle, the one from the other.

This may be done nearly by either of the four following proportions,

viz. As 7 is to 22, so is the diameter to the circumference.
Or, As 1 is to 3 1416, so is the diam. to the circumf.
Or, As 113 to 355, so is the diam. to the circumf.*
And, as 1 : 318309 :: the circumf. : the diameter.

* For let ABCD be any circle, whose centre is E, and let AB, BC be any two equal arcs. Draw the several chords as in the figure, and join BE; also draw the diameter DA, which produce to F, till BF be equal to the chord BD.



Then the two isosceles triangles DEB, DBF, are equiangular, because they have the angle at D common; consequently DE : DB :: DB : DF. But the two triangles AFB, DCB are identical, or equal in all respects, because they have the angle F = the angle BDC, being each equal to the angle ADB, these being subtended by the equal arcs AB, BC; also the exterior angle FAB of the quadrangle ABCD, is equal to the opposite interior angle at C; and the two triangles have also the side BF = the side BD; therefore the side AF is also equal to the side DC. Hence the proportion above, viz. DE : DB :: DB : DF = DA + AF, becomes DE : DB :: DB : 2DE + DC. Then, by taking the rectangles of the extremes and means, it is $DB^2 = 2DE^2 + DE \cdot DC$.

Now, if the radius DE be taken = 1, this expression becomes $DB^2 = 2 + DC$, and hence the root $DB = \sqrt{2 + DC}$. That is, if the measure of the supplemental chord of any arc be increased by the number 2, the square root of the sum will be the supplemental chord of half that arc.

Now, to apply this to the calculation of the circumference of

Ex. 1. To find the circumference of the circle whose diameter is 20.

By the first rule, as $7 : 22 :: 20 : 62\frac{6}{7}$, the answer.

Ex. 2. If the circumference of the earth be 24877·4 miles what is its diameter?

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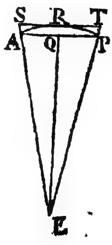
the circle, let the arc AC be taken equal to $\frac{1}{6}$ of the circumference, and be successively bisected by the above theorem: thus the chord AC of $\frac{1}{6}$ of the circumference, is the side of the inscribed regular hexagon, and is therefore equal to the radius AE or 1: hence, in the right-angled triangle ACD, it will be DC $= \sqrt{AD^2 - AC^2} = \sqrt{2^2 - 1^2} = \sqrt{3} = 1·7320508076$, the supplemental chord of $\frac{1}{6}$ of the periphery.

Then, by the foregoing theorem, by always bisecting the arcs, and adding 2 to the last square root, there will be found the supplemental chords of the 12th, the 24th, the 48th, the 96th, &c., parts of the periphery; thus,

$$\left.\begin{array}{l} \sqrt{3·7320508076} = 1·9318516525 \\ \sqrt{3·9318516525} = 1·9828897227 \\ \sqrt{3·9828897227} = 1·9957178465 \\ \sqrt{3·9957178465} = 1·9989291743 \\ \sqrt{3·9989291743} = 1·9997322757 \\ \sqrt{3·9997322757} = 1·9999330678 \\ \sqrt{3·9999330678} = 1·9999832669 \\ \sqrt{3·9999832669} = \text{- - - - - - - -} \end{array}\right\} \begin{array}{c}\text{for the supplemen-}\\ \text{tal chord of}\end{array} \left\{\begin{array}{c} \frac{1}{12} \\ \frac{1}{24} \\ \frac{1}{48} \\ \frac{1}{96} \\ \frac{1}{192} \\ \frac{1}{384} \\ \frac{1}{768} \\ \frac{1}{1536} \end{array}\right\} \text{of the periphery.}$$

Since then it is found that 3·9999832669 is the square of the supplemental chord of the 1536th part of the periphery, let this number be taken from 4, which is the square of the diameter, and the remainder 0·0000167331 will be the square of the chord of the said 1536th part of the periphery, and consequently the root $\sqrt{0·0000167331} = 0·0040906112$ is the length of that chord; this number then being multiplied by 1536 gives 6·2831788 for the perimeter of a regular polygon of 1536 sides inscribed in the circle; which, as the sides of the polygon nearly coincide with the circumference of the circle, must also express the length of the circumference itself, very nearly.



But now, to show how near this determination is to the truth, let AQP = 0·0040906112 represent one side of such a regular polygon of 1536 sides, and SRT a side of another similar polygon described about the circle; and from the centre E let the perpendicular EQR be drawn, bisecting AP and ST in Q and R. Then since AQ is $= \frac{1}{2}$ AP $=$ 0·0020453056, and EA $= 1$, therefore $EQ^2 = EA^2 - AQ^2 = ·9999958167$, and consequently its root gives EQ $= ·9999979084$; then because of the

By the 2d rule, as 3·1416 : 1 : : 24877·4 : 7918·7 nearly the diameter.

By the 3d rule, as 355 : 113 : : 24877·4 : 7918·7 nearly.

By the 4th rule, as 1 : ·318309 : : 24877·4 : 7918·7 nearly.

PROBLEM VIII.

To find the Length of any Arc of a Circle.

MULTIPLY the decimal ·017453 by the degrees in the given arc, and that product by the radius of the circle, for the length of the arc*.

Ex. 1. To find the length of an arc of 30 degrees, the radius being 9 feet. Ans. 4·7115.

Ex. 2. To find the length of an arc of 12° 10', or $12°\frac{1}{6}$, the radius being 10 feet. Ans. 2·1231.

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parallels AP, ST, it is EQ : ER . . AP : ST : : as the whole inscribed perimeter: to the circumscribed one, that is, as ·9999979084 : 1 : : 6·2831788 . 6·2831920 the perimeter of the circumscribed polygon. Now, the circumference of the circle being greater than the perimeter of the inner polygon, but less than that of the outer, it must consequently be greater than 6·2831788, but less than 6·2831920, and must therefore be nearly equal to $\frac{1}{2}$ their sum, or 6·2831854, which in fact is true to the last figure, which should be a 3, instead of the 4.

Hence the circumference being 6·2831854 when the diameter is 2, it will be the half of that, or 3·1415927, when the diameter is 1, to which the ratio in the rule, viz. 1 to 3·1416, is very near. Also the first ratio in the rule, 7 to 22 or 1 to $3\frac{1}{7}$ = 3·1428 &c. is another near approximation. But the third ratio, 113 to 355, = 1 to 3·1415929, is the nearest.

* It having been found, in the demonstration of the foregoing problem, that when the radius of a circle is 1, the length of the whole circumference is 6·2831854, which consists of 360 degrees; therefore as 360° : 6·2831854 : : 1° : ·017453, &c. the length of the arc of 1 degree. Hence the decimal ·017453 multiplied by any number of degrees, will give the length of the arc of those degrees. And because the circumferences and arcs are in proportion as the diameters, or as the radii of the circles, therefore as the radius 1 is to any other radius r, so is the length of the arc above mentioned, to ·017453 × degrees in the arc × r, which is the length of that arc, as in the rule.

PROBLEM IX.

*To find the Area of a Circle**.

RULE I. MULTIPLY half the circumference by half the diameter. Or multiply the whole circumference by the whole diameter, and take ¼ of the product.

RULE II. Square the diameter, and multiply that square by the decimal ·7854, for the area.

RULE III. Square the circumference, and multiply that square by the decimal ·07958.

Ex. 1. To find the area of a circle whose diameter is 10, and its circumference 31·416.

By Rule 1.	By Rule 2.	By Rule 3.
31·416	·7854	31·416
10	$10^2 =$ 100	31·416
4) 314·16	78·54	986·965
78·54		·07958
		78·54

So that the area is 78·54 by all the three rules.

Ex. 2. To find the area of a circle, whose diameter is 7, and circumference 22. Ans. $38\frac{1}{2}$.

Ex. 3. How many square yards are in a circle, whose diameter is $3\frac{1}{2}$ feet? Ans. 1·069.

Ex. 4. To find the area of a circle, whose circumference is 12 feet. Ans. 11·4595.

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* The first rule is proved in the Geom. theor. 94.

And the 2d and 3rd rules are deduced from the first rule, in this manner.—By that rule, $dc \div 4$ is the area, when d denotes the diameter, and c the circumference. But, by prob. 7, c is $= 3{\cdot}1416d$; therefore the said area $dc \div 4$, becomes $d \times 3{\cdot}1416d \div 4 = {\cdot}7854d^2$, which gives the 2d rule.—Also by the same prob. 7, d is $= c \div 3{\cdot}1416$; therefore again the same first area $dc \div 4$, becomes $c \div 3{\cdot}1416 \times c \div 4 = c^2 \div 12.5664$, which is $= c^2 \times {\cdot}07958$, by taking the reciprocal of 12·5664, or changing that divisor into the multiplier ·07958; which gives the 3d rule.

Corol. Hence the areas of different circles are in proportion to one another, as the square of their diameters or as the square of their circumferences; as before proved in the Geom. theor. 93.

PROBLEM X.

To find the Area of a Circular Ring, or of the Space included between the Circumferences of two Circles; the one being contained within the other.

TAKE the difference between the areas of the two circles, as found by the last problem, for the area of the ring.—Or, which is the same thing, subtract the square of the less diameter from the square of the greater, and multiply their difference by ·7854.—Or lastly, multiply the sum of the diameters by the difference of the same, and that product by ·7854; which is still the same thing, because the product of the sum and difference of any two quantities, is equal to the difference of their squares.

Ex. 1. The diameters of two concentric circles being 10 and 6, required the area of the ring contained between their circumferences.

Here $10 + 6 = 16$ the sum, and $10 - 6 = 4$ the diff.
Therefore $·7854 \times 16 \times 4 = ·7854 \times 64 = 50·2656$, the area.

Ex. 2. What is the area of the ring, the diameters of whose bounding circles are 10 and 20? Ans. 235·62.

PROBLEM XI.

To find the Area of the Sector of a Circle.

RULE I. MULTIPLY the radius, or half the diameter, by half the arc of the sector, for the area. Or, multiply the whole diameter by the whole arc of the sector, and take $\frac{1}{4}$ of the product. The reason of which is the same as for the first rule to problem 9, for the whole circle.

RULE II. Compute the area of the whole circle: then say, as 360 is to the degrees in the arc of the sector, so is the area of the whole circle, to the area of the sector.

This is evident, because the sector is proportional to the length of the arc, or to the degrees contained in it.

Ex. 1. To find the area of a circular sector, whose arc contains 18 degrees; the diameter being 3 feet.

1. By the 1st Rule.

First, $3·1416 \times 3 = 9·4248$, the circumference.
And $360 : 18 :: 9·4248 : ·47124$, the length of the arc.
Then $·47124 \times 3 \div 4 = 1·41372 \div 4 = ·35343$, the area.

2. By the 2d Rule.

First, $\cdot 7854 \times 3^2 = 7\cdot 0686$, the area of the whole circle.

Then, as $360 : 18 :: 7\cdot 0686 : \cdot 35343$, the area of the sector.

Ex. 2. To find the area of a sector, whose radius is 10, and arc 20. Ans. 100.

Ex. 3. Required the area of a sector, whose radius is 25, and its arc containing 147° 29′. Ans. 804·3986.

PROBLEM XII.

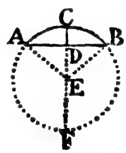
To find the Area of a Segment of a Circle.

RULE I. FIND the area of the sector having the same arc with the segment, by the last problem.

Find also the area of the triangle, formed by the chord of the segment and the two radii of the sector.

Then add these two together for the answer, when the segment is greater than a semicircle: or subtract them when it is less than a semicircle.—As is evident by inspection.

Ex. 1. To find the area of the segment ACBDA, its chord AB being 12, and the radius AE or CE 10.



FIRST, As AE : sin. $\angle$ D 90° :: AD : sin. 36° $52'\frac{1}{5}$ = 36·87 degrees, the degrees in the $\angle$ AEC or arc AC. Their double, 73·74, are the degrees in the whole arc ACB.

Now $\cdot 7854 \times 400 = 314\cdot 16$, the area of the whole circle.

Therefore $360° : 73\cdot 74 :: 314\cdot 16 : 64\cdot 3504$, area of the sector ACBE.

Again, $\sqrt{AE^2 - AD^2} = \sqrt{100 - 36} = \sqrt{64} = 8 =$ DE.

Theref. AD $\times$ DE $= 6 \times 8 = 48$, the area of the triangle AEB.

Hence sector ACBE $-$ triangle AEB $= 16\cdot 3504$, area of seg. ACBDA.

RULE II. Multiply the square of the radius of the circle by either half the difference of the arc ACB and its sine (both to the radius 1), or half the sum of the arc and its sine, according as the segment is less or greater than a semicircle; the product will be the area.

The reason of this rule, also, is evident from an inspection of the diagram.

EXAM. the same as before, in which AB = 12, AE = 10; and from the former computation arc ACB = 73° $44'\frac{2}{5}$.

Then, by Hutton's Mathematical Tables, pp. 340, &c.

arc $73^\circ\ 44'\frac{2}{5}$, to radius $1 = 1{\cdot}2870059$
sin. $73^\circ\ 44'\frac{2}{5}$, to radius $1 = {\cdot}9600010$

$$2)\ {\cdot}3270069$$

$$\cdot 1635034$$

whence, $\cdot 1635034 \times 10^2 = 16{\cdot}35034$, the area of the segment; very nearly as before.

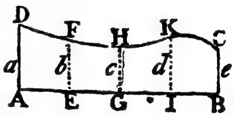
Ex. 2. What is the area of the segment, whose height is 18, and diameter of the circle 50? Ans. 636·375.

Ex. 3. Required the area of the segment whose chord is 16, the diameter being 20? Ans. 44·728.

PROBLEM XIII.

To measure long Irregular Figures.

TAKE or measure the breadth at both ends, and at several places, at equal distances. Then add together all these intermediate breadths and half the two extremes, which sum multiply by the length, and divide by the number of parts, for the area *.



* This rule is made out as follows:—Let ABCD be the irregular piece; having the several breadths AD, EF, GH, IK, BC, at the equal distances AE, EG, GI, IB. Let the several breadths in order be denoted by the corresponding letters a, b, c, d, e, and the whole length AB by l; then compute the areas of the parts into which the figure is divided by the perpendiculars, as so many trapezoids, by prob. 3, and add them all together. Thus, the sum of the parts is,

$$\frac{a+b}{2} \times \text{AE} + \frac{b+c}{2} \times \text{EG} + \frac{c+d}{2} \times \text{GI} + \frac{d+e}{2} \times \text{IB}$$

$$= \frac{a+b}{2} \times \tfrac{1}{4}l + \frac{b+c}{2} \times \tfrac{1}{4}l + \frac{c+d}{2} \times \tfrac{1}{4}l + \frac{d+e}{2} \times \tfrac{1}{4}l$$

$$= (\tfrac{1}{2}a + b + c + d + \tfrac{1}{2}e) \times \tfrac{1}{4}l = (m + b + c + d)\tfrac{1}{4}l,$$

which is the whole area, agreeing with the rule: m being the arithmetical mean between the extremes, or half the sum of them both, and 4 the number of the parts. And the same for any other number of parts whatever.

Note. If the perpendiculars or breadths be not at equal distances, compute all the parts separately, as so many trapezoids, and add them all together for the whole area.

Or else, add all the perpendicular breadths together, and divide their sum by the number of them for the mean breadth, to multiply by the length; which will give the whole area, not far from the truth.

Ex. 1. The breadths of an irregular figure, at five equidistant places, being 8·2, 7·4, 9·2, 10·2, 8·6; and the whole length 39; required the area.

8·2	3·52 sum.
8·6	39
2) 16·8 sum of the extremes.	3168
8·4 mean of the extremes.	1056
7·4	4) 1372·8
9·2	343·2 Ans.
10·2	
35·2 sum.	

Ex. 2. The length of an irregular figure being 84, and the breadths at six equidistant places 17·4, 20·6, 14·2, 16·5, 20·1, 24·4; what is the area? Ans. 1550·64.

PROBLEM XIV.

To find the Area of an Ellipsis or Oval.

Multiply the longest diameter, or axis, by the shortest; then multiply the product by the decimal ·7854, for the area. As appears from cor. 2, theor. 3, of the Ellipse, in the Conic Sections.

Ex. 1. Required the area of an ellipse whose two axes are 70 and 50. Ans. 2748·9.

Ex. 2. To find the area of the oval whose two axes are 24 and 18. Ans. 339·2928.

PROBLEM XV.

To find the Area of an Elliptic Segment.

Find the area of a corresponding circular segment, having the same height and the same vertical axis or diameter. Then

say, as the said vertical axis is to the other axis, parallel to the segment's base, so is the area of the circular segment before found, to the area of the elliptic segment sought. This rule also comes from cor. 2, theor. 3, of the Ellipse.

Ex. 1. To find the area of the elliptic segment, whose height is 20, the vertical axis being 70, and the parallel axis 50. Ans. 648·13.

Ex. 2. Required the area of an elliptic segment, cut off parallel to the shorter axis; the height being 10, and the two axes 25 and 35. Ans. 162·03.

Ex. 3. To find the area of the elliptic segment, cut off parallel to the longer axis; the height being 5, and the axes 25 and 35. Ans. 97·8125.

PROBLEM XVI.

To find the Area of a Parabola, or its Segment.

Multiply the base by the perpendicular height; then take two-thirds of the product for the area. As is proved in theorem 17 of the Parabola, in the Conic Sections.

Ex. 1. To find the area of a parabola; the height being 2, and the base 12.

Here $2 \times 12 = 24$. Then $\frac{2}{3}$ of $24 = 16$, is the area.

Ex. 2. Required the area of the parabola, whose height is 10, and its base 16. Ans. $106\frac{2}{3}$.

MENSURATION OF SOLIDS.

By the Mensuration of Solids are determined the spaces included by contiguous surfaces; and the sum of the measures of these including surfaces is the whole surface or superficies of the body.

The measure of a solid, is called its solidity, capacity, or content.

Solids are measured by cubes, whose sides are inches, or feet, or yards, &c. And hence the solidity of a body is said to be so many cubic inches, feet, yards, &c. as will fill its capacity or space, or another of an equal magnitude.

The least solid measure is the cubic inch, other cubes being taken from it according to the proportion in the fol-

lowing table, which is formed by cubing the linear proportions.

Table of Cubic or Solid Measures.

1728	cubic inches make	1 cubic foot
27	cubic feet -	1 cubic yard
166	cubic yards -	1 cubic pole
64000	cubic poles -	1 cubic furlong
512	cubic furlongs	1 cubic mile.

PROBLEM I.

To find the Superficies of a Prism or Cylinder.

MULTIPLY the perimeter of one end of the prism by the length or height of the solid, and the product will be the surface of all its sides. To which add also the area of the two ends of the prism, when required *.

Or, compute the areas of all the sides and ends separately, and add them all together.

Ex. 1. To find the surface of a cube, the length of each side being 20 feet. Ans. 2400 feet.

Ex. 2. To find the whole surface of a triangular prism, whose length is 20 feet, and each side of its end or base 18 inches. Ans. 91·948 feet.

Ex. 3. To find the convex surface of a round prism, or cylinder, whose length is 20 feet, and the diameter of its base is 2 feet. Ans. 125·664.

Ex. 4. What must be paid for lining a rectangular cistern with lead, at *3d.* a pound weight, the thickness of the lead being such as to weigh 7lb. for each square foot of surface; the inside dimensions of the cistern being as follow, viz. the length 3 feet 2 inches, the breadth 2 feet 8 inches, and depth 2 feet 6 inches? Ans. *3l.* *5s.* $9\frac{1}{2}d.$

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* The truth of this will easily appear, by considering that the sides of any prism are parallelograms, whose common length is the same as the length of the solid, and their breadths taken all together make up the perimeter of the ends of the same.

And the rule is evidently the same for the surface of a cylinder.

PROBLEM II.

To find the Surface of a Pyramid or Cone.

MULTIPLY the perimeter of the base by the slant height, or length of the side, and half the product will evidently be the surface of the sides, or the sum of the areas of all the triangles which form it. To which add the area of the end or base, if requisite.

Ex. 1. What is the upright surface of a triangular pyramid, the slant height being 20 feet, and each side of the base 3 feet? Ans. 90 feet.

Ex. 2. Required the convex surface of a cone, or circular pyramid, the slant height being 50 feet, and the diameter of its base $8\frac{1}{2}$ feet. Ans. 667·59.

PROBLEM III.

To find the Surface of the Frustum of a Pyramid or Cone, being the lower part, when the top is cut off by a plane parallel to the base.

ADD together the perimeters of the two ends, and multiply their sum by the slant height, taking half the product for the answer.—As is evident, because the sides of the solid are trapezoids, having the opposite sides parallel.

Ex. 1. How many square feet are in the surface of the frustum of a square pyramid, whose slant height is 10 feet; also each side of the base or greater end being 3 feet 4 inches, and each side of the less end 2 feet 2 inches? Ans. 110 feet.

Ex. 2. To find the convex surface of the frustum of a cone, the slant height of the frustum being $12\frac{1}{2}$ feet, and the circumferences of the two ends 6 and 8·4 feet.

Ans. 90 feet.

PROBLEM IV.

To find the Solid Content of any Prism or Cylinder.

FIND the area of the base, or end, whatever the figure of it may be; and multiply it by the length of the prism or cylinder, for the solid content *.

* This rule appears from the Geom. theor. 110, cor. 2. The same is more particularly shown as follows: Let the annexed

Note. For a cube, take the cube of its side by multiplying this twice by itself; and for a parallelopipedon, multiply the length, breadth, and depth all together, for the content.

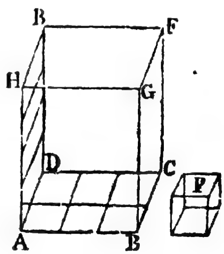
Ex. 1. To find the solid content of a cube, whose side is 24 inches. Ans. 13824.

Ex. 2. How many cubic feet are in a block of marble, its length being 3 feet 2 inches, breadth 2 feet, 8 inches, and thickness 2 feet 6 inches? Ans. $21\frac{1}{9}$.

Ex. 3. How many gallons of water will the cistern contain, whose dimensions are the same as in the last example, when 282 cubic inches are contained in one gallon? Ans. $129\frac{17}{47}$.

Ex. 4. Required the solidity of a triangular prism, whose length is 10 feet, and the three sides of its triangular end or base are 3, 4, 5 feet. Ans. 60.

Ex. 5. Required the content of a round pillar, or cylinder, whose length is 20 feet, and circumference 5 feet 6 inches. Ans. 48·1459 feet.



rectangular parallelopipedon be the solid to be measured, and the cube P the solid measuring unit, its side being 1 inch, or 1 foot, &c.; also, let the length and breadth of the base ABCD, and also the height AH, be each divided into spaces equal to the length of the base of the cube P, namely, here 3 in the length and 2 in the breadth, making 3 times 2 or 6 squares in the base AC, each equal to the base of the cube P. Hence it is manifest that the parallelopipedon will contain the cube P, as many times as the base AC contains the base of the cube, repeated as often as the height AH contains the height of the cube. That is, the content of any parallelopipedon is found, by multiplying the area of the base by the altitude of that solid.

And because all prisms and cylinders are equal to parallelopipedons of equal bases and altitudes, by Geom. theor. 108, it follows that the rule is general for all such solids, whatever the figure of the base may be.

PROBLEM V.

To find the Content of any Pyramid or Cone.

FIND the area of the base, and multiply that area by the perpendicular height; then take $\frac{1}{3}$ of the product for the content *.

Ex. 1. Required the solidity of a square pyramid, each side of its base being 30, and its perpendicular height 25.
Ans. 7500.

Ex. 2. To find the content of a triangular pyramid, whose perpendicular height is 30, and each side of the base 3.
Ans. 38·971143.

Ex. 3. To find the content of a triangular pyramid, its height being 14 feet 6 inches, and the three sides of its base 5, 6, 7 feet. Ans. 71·0352.

Ex. 4. What is the content of a pentagonal pyramid, its height being 12 feet, and each side of its base 2 feet?
Ans. 27·5276.

Ex. 5. What is the content of the hexagonal pyramid, whose height is 6·4 feet, and each side of its base 6 inches?
Ans. 1·38564 feet.

Ex. 6. Required the content of a cone, its height being $10\frac{1}{2}$ feet, and the circumference of its base 9 feet.
Ans. 22·56093.

PROBLEM VI.

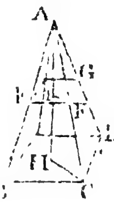
To find the Solidity of a Frustum of a Cone or Pyramid.

ADD into one sum, the areas of the two ends, and the mean proportional between them: and take $\frac{1}{3}$ of that sum for a mean area; which being multiplied by the perpen-

* This rule follows from that of the prism, because any pyramid is $\frac{1}{3}$ of a prism of equal base and altitude; by Geom. theor. 115, cor. 1 and 2.

dicular height, or length of the frustum, will give its content *.

Note. This general rule may be otherwise expressed, as follows, when the ends of the frustum are circles or regular polygons. In this latter case, square one side of each polygon, and also multiply the one side by the other; add all these three products together; then multiply their sum by the tabular area proper to the polygon, and take one-third of the product for the mean area, to be multiplied by the length, to give the solid content. And in the case of the frustum of a cone, the ends being circles, square the diameter or the circumference of each end, and also multiply the same two dimensions together; then take the sum of the three products, and multiply it by the proper tabular number, viz. by ·7854 when the diameters are used, or by ·07958 in using the circumferences; then taking one-third of the product, to multiply by the length, for the content.



* Let ABCD be any pyramid, of which BCDGFE is a frustum. And put a^2 for the area of the base BCD, b^2 the area of the top EFG, h the height IH of the frustum, and c the height AI of the top part above it. Then $c + h =$ AH is the height of the whole pyramid.

Hence, by the last prob. $\frac{1}{3}a^2(c + h)$ is the content of the whole pyramid ABCD, and $\frac{1}{3}b^2c$ the content of the top part AEFG; therefore the difference $\frac{1}{3}a^2(c + h) - \frac{1}{3}b^2c$ is the content of the frustum BCDGFE. But the quantity c being no dimension of the frustum, it must be expelled from this formula, by substituting its value, found in the following manner. By Geom. theor. 112, $a^2 : b^2 :: (c + h)^2 : c^2$, or $a : b :: c + h : c$, hence (Geom. th. 69) $a - b : b :: h : c$, and $a - b : a :: h : c + h$; hence therefore $c = \frac{bh}{a-b}$, and $c + h = \frac{ah}{a-b}$; then these values of c and $c + h$ being substituted for them in the expression for the content of the frustum gives that content

$$= \tfrac{1}{3}a^2 \times \frac{ah}{a-b} - \tfrac{1}{3}b^2 \times \frac{bh}{a-b} = \tfrac{1}{3}h \times \frac{a^3 - b^3}{a-b} = \tfrac{1}{3}h \times (a^2 + ab + b^2);$$

which is the rule above given; ab being the mean between a^2 and b^2.

Note. If D, d, be the corresponding linear dimensions of the ends, δ their difference, m the appropriate multiplier, h the height of the frustum, then is the content $= \frac{1}{3}mh(3\text{D}d + \delta)$; which is a convenient practical expression.

Ex. 1. To find the number of solid feet in a piece of timber, whose bases are squares, each side of the greater end being 15 inches, and each side of the less end 6 inches; also, the length or the perpendicular altitude 24 feet.

Ans. $19\frac{1}{2}$.

Ex. 2. Required the content of a pentagonal frustum, whose height is 5 feet, each side of the base 18 inches; and each side of the top or less end 6 inches. Ans. 9·31925 feet.

Ex. 3. To find the content of a conic frustum, the altitude being 18, the greatest diameter 8, and the least diameter 4. Ans. 527·7888.

Ex. 4. What is the solidity of the frustum of a cone, the altitude being 25, also the circumference at the greater end being 20, and at the less end 10? Ans. 464·216.

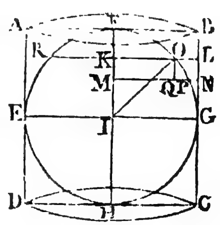
Ex. 5. If a cask, which is two equal conic frustums joined together at the bases, have its bung diameter 28 inches, the head diameter 20 inches, and length 40 inches; how many gallons of wine will it hold? Ans. 79·0613.

PROBLEM VII.

To find the Surface of a Sphere, or any Segment.

Rule I. Multiply the circumference of the sphere by its diameter, and the product will be the whole surface of it *.

* These rules come from the following theorems for the surface of a sphere, viz. That the said surface is equal to the curve surface of its circumscribing cylinder; or that it is equal to 4 great circles of the same sphere, or of the same diameter; which are thus proved.



Let ABCD be a cylinder, circumscribing the sphere EFGH; the former generated by the rotation of the rectangle FBCH about the axis or diameter FH; and the latter by the rotation of the semicircle FGH about the same diameter FH. Draw two lines KL, MN, perpendicular to the axis, intercepting the parts LN, OP, of the cylinder and sphere; then will the ring or cylindric surface generated by the rotation of LN, be equal to the ring or spherical surface generated by the arc OP. For, first, suppose the parallels KL and MN to be indefinitely near together; drawing IO, and also OQ parallel

RULE II. Square the diameter and multiply that square by 3·1416, for the surface.

RULE III. Square the circumference; then either multiply that square by the decimal ·3183, or divide it by 3·1416, for the surface.

Note. For the surface of a segment or frustum, multiply the whole circumference of the sphere by the height of the part required.

Ex. 1. Required the convex superficies of a sphere, whose diameter is 7, and circumference 22. Ans. 154.

Ex. 2. Required the superficies of a globe, whose diameter is 24 inches. Ans. 1809·5616.

Ex. 3. Required the area of the whole surface of the earth, its diameter being $7957\frac{3}{4}$ miles, and its circumference 25000 miles. Ans. 198943750 sq. miles.

Ex. 4. The axis of a sphere being 42 inches, what is the convex superficies of the segment whose height is 9 inches? Ans. 1187·5248 inches.

Ex. 5. Required the convex surface of a spherical zone, whose breadth or height is 2 feet, and cut from a sphere of $12\frac{1}{2}$ feet diameter. Ans. 78·54 feet.

to LN. Then the two triangles IKO, OQP, being equiangular, it is, as OP : OQ or LN : : IO or KL : KO : : circumference described by KL : circumf. described by KO; therefore the rectangle OP × circumf. of KO is equal to the rectangle LN × circumf. of KL; that is, the ring described by OP on the sphere, is equal to the ring described by LN on the cylinder.

And as this is every where the case, therefore the sums of any corresponding number of these are also equal; that is, the whole surface of the sphere, described by the whole semicircle FGH, is equal to the whole curve surface of the cylinder, described by the height BC; as well as the surface of any segment described by FO, equal to the surface of the corresponding segment described by BL.

Corol. 1. Hence the surface of the sphere is equal to 4 of its great circles, or equal to the circumference EFGH, or of DC, multiplied by the height BC, or by the diameter FH.

Corol. 2. Hence also, the surface of any such part as a segment or frustum, or zone, is equal to the same circumference of the sphere, multiplied by the height of the said part. And consequently such spherical curve surfaces are to one another in the same proportion as their altitudes.

PROBLEM VIII.

To find the Solidity of a Sphere or Globe.

RULE I. Multiply the surface by the diameter, and take $\frac{1}{6}$ of the product for the content *. Or, which is the same thing, multiply the square of the diameter by the circumference, and take $\frac{1}{6}$ of the product.

RULE II. Take the cube of the diameter, and multiply it by the decimal ·5236, for the content.

RULE III. Cube the circumference, and multiply by ·01688.

Ex. 1. To find the solid content of the globe of the earth, supposing its circumference to be 25000 miles.

Ans. 263750000000 miles.

Ex. 2. Supposing that a cubic inch of cast iron weighs ·269 of a lb. averd. what is the weight of an iron ball of 5·04 inches diameter?

PROBLEM IX.

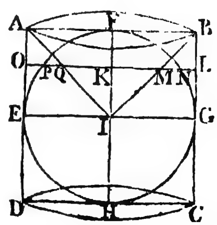
To find the Solid Content of a Spherical Segment.

† RULE I. From 3 times the diameter of the sphere

* For, put d = the diameter, c = the circumference, and s = the surface of the sphere, or of its circumscribing cylinder; also, a = the number 3·1416.

Then, $\frac{1}{4}s$ is = the base of the cylinder, or one great circle of the sphere; and d is the height of the cylinder; therefore $\frac{1}{4}ds$ is the content of the cylinder. But $\frac{2}{3}$ of the cylinder is the sphere, by th. 117, Geom. that is, $\frac{2}{3}$ of $\frac{1}{4}ds$, or $\frac{1}{6}ds$ is the sphere; which is the first rule.

Again, because the surface s is $= ad^2$; therefore $\frac{1}{6}ds = \frac{1}{6}ad^3 = \cdot5236d^3$, is the content, as in the 2d rule. Also, d being $= c \div a$, therefore $\frac{1}{6}ad^3 = \frac{1}{6}c^3 \div a^2 = \cdot01688$, the 3d rule for the content.



† By corol. 3, of theor. 117, Geom. it appears that the spheric segment PFN, is equal to the difference between the cylinder ABLO, and the conic frustum ABMQ.

But, putting d = AB or FH the diameter of the sphere or cylinder, h = FK the height of the segment, r = PK the radius of its base, and a = 3·1416; then the content of the cone ABI is $= \frac{1}{4}ad^2 \times \frac{1}{2}$FI $= \frac{1}{24}ad^3$; and by the similar cones ABI, QMI, as

take double the height of the segment; then multiply the remainder by the square of the height, and the product by the decimal ·5236, for the content.

RULE II. To 3 times the square of the radius of the segment's base, add the square of its height; then multiply the sum by the height, and the product by ·5236, for the content.

Ex. 1. To find the content of a spherical segment, of 2 feet in height, cut from a sphere of 8 feet diameter.

Ans. 41·888.

Ex. 2. What is the solidity of the segment of a sphere, its height being 9, and the diameter of its base 20?

Ans. 1795·4244.

Note. The general rules for measuring the most useful figures having been now delivered, we may proceed to apply them to the several practical uses in life, as follows.

$FI^3 : KI^3 :: \frac{1}{24}ad^3 : \frac{1}{24}ad^3 \times (\frac{\frac{1}{2}d - h}{\frac{1}{2}d})^3$ = the cone QMI; therefore the cone ABI — the cone QMI $= \frac{1}{24}ad^3 - \frac{1}{24}ad^3 \times (\frac{\frac{1}{2}d-h}{\frac{1}{2}d})^3 = \frac{1}{4}ad^2h - \frac{1}{2}adh^2 + \frac{1}{3}ah^3$ is = the conic frustum ABMQ.

And $\frac{1}{4}ad^2h$ is = the cylinder ABLO.

Then the difference of these two is $\frac{1}{2}adh^2 - \frac{1}{3}ah^3 = \frac{1}{6}ah^2 \times (3d - 2h)$, for the spheric segment PFN; which is the first rule.

Again, because $PK^2 = FK \times KH$ (cor. to theor. 87, Geom.) or $r^2 = h(d - h)$, therefore $d = \frac{r^2}{h} + h$, and $3d - 2h = \frac{3r^2}{h} + h = \frac{3r^2+h^2}{h}$; which being substituted in the former rule, it becomes $\frac{1}{6}ah^2 \times \frac{3r^2 + h^2}{h} = \frac{1}{6}ah \times (3r^2 + h^2)$, which is the 2d rule.

Note. By subtracting a segment from a half sphere, or from another segment, the content of any frustum or zone may be found.

LAND SURVEYING.

SECTION I.

DESCRIPTION AND USE OF THE INSTRUMENTS.

1. OF THE CHAIN.

LAND is measured with a chain, called Gunter's Chain, from its inventor, the length of which is 4 poles, or 22 yards, or 66 feet. It consists of 100 equal links; and the length of each link is therefore $\frac{22}{100}$ of a yard, or $\frac{66}{100}$ of a foot, or 7·92 inches.

Land is estimated in acres, roods, and perches. An acre is equal to 10 square chains, or as much as 10 chains in length and 1 chain in breadth. Or, in yards, it is $220 \times 22 = 4840$ square yards. Or, in poles, it is $40 \times 4 = 160$ square poles. Or, in links, it is $1000 \times 100 = 100000$ square links: these being all the same quantity.

Also, an acre is divided into 4 parts called roods, and a rood into 40 parts called perches, which are square poles, or the square of a pole of $5\frac{1}{2}$ yards long, or the square of $\frac{1}{4}$ of a chain, or of 25 links, which is 625 square links. So that the divisions of land measure will be thus:

625 sq. links = 1 pole or perch
40 perches = 1 rood
4 roods = 1 acre.

The lengths of lines, measured with a chain, are best set down in links as integers, every chain in length being 100 links; and not in chains and decimals. Therefore, after the content is found, it will be in square links; then cut off five of the figures on the right hand for decimals, and the rest will be acres. These decimals are then multiplied by 4 for roods, and the decimals of these again by 40 for perches.

EXAM. Suppose the length of a rectangular piece of ground be 792 links, and its breadth 385; to find the area in acres, roods, and perches.

792	3·04920
385	4
3960	·19680
6336	40
2376	
	7·87200
3·04920	

Ans. 3 acres, 0 roods, 7 perches.

2. OF THE PLAIN TABLE.

This instrument consists of a plain rectangular board, of any convenient size: the centre of which, when used, is fixed by means of screws to a three-legged stand, having a ball and socket, or other joint, at the top, by means of which, when the legs are fixed on the ground, the table is inclined in any direction.

To the table belong various parts, as follow.

1. A frame of wood, made to fit round its edges, and to be taken off, for the convenience of putting a sheet of paper on the table. One side of this frame is usually divided into equal parts, for drawing lines across the table, parallel or perpendicular to the sides; and the other side of the frame is divided into 360 degrees, to a centre in the middle of the table; by means of which the table may be used as a theodolite, &c.

2. A magnetic needle and compass, either screwed into the side of the table, or fixed beneath its centre, to point out the directions, and to be a check on the sights.

3. An index, which is a brass two-foot scale, with either a small telescope, or open sights set perpendicularly on the ends. These sights and one edge of the index are in the same plane, and that is called the fiducial edge of the index.

To use this instrument, take a sheet of paper which will cover it, and wet it to make it expand; then spread it flat on the table, pressing down the frame on the edges, to stretch it and keep it fixed there; and when the paper is become dry, it will, by contracting again, stretch itself smooth and flat from any cramps and unevenness. On this paper is to be drawn the plan or form of the thing measured.

Thus, begin at any proper part of the ground, and make a point on a convenient part of the paper or table, to represent that place on the ground; then fix in that point one leg of the compasses, or a fine steel pin, and apply to it the fiducial edge of the index, moving it round till through the sights you perceive some remarkable object, as the corner of a field, &c.; and from the station-point draw a line with the point of the compasses along the fiducial edge of the index, which is called setting or taking the object: then set another object or corner, and draw its line; do the same by another; and so on, till as many objects are taken as may be thought fit. Then measure from the station towards as many of the objects as may be necessary, but not more, taking the requisite offsets to corners or crooks in the hedges, laying the measures down on their respective lines on the table. Then

at any convenient place measured to, fix the table in the same position, and set the objects which appear from that place; and so on, as before. And thus continue till the work is finished, measuring such lines only as are necessary, and determining as many as may be by intersecting lines of direction drawn from different stations.

Of shifting the Paper on the Plain Table.

When one paper is full, and there is occasion for more, draw a line in any manner through the farthest point of the last station line, to which the work can be conveniently laid down; then take the sheet off the table, and fix another on, drawing a line over it, in a part the most convenient for the rest of the work; then fold or cut the old sheet by the line drawn on it, applying the edge to the line on the new sheet, and, as they lie in that position, continue the last station line on the new paper, placing on it the rest of the measure, beginning at where the old sheet left off. And so on from sheet to sheet.

When the work is done, and you would fasten all the sheets together into one piece, or rough plan, the aforesaid lines are to be accurately joined together, in the same manner as when the lines were transferred from the old sheets to the new ones. But it is to be noted, that if the said joining lines, on the old and new sheets, have not the same inclination to the side of the table, the needle will not point to the original degree when the table is rectified; and if the needle be required to respect still the same degree of the compass, the easiest way of drawing the line in the same position, is to draw them both parallel to the same sides of the table, by means of the equal divisions marked on the other two sides.

3. OF THE THEODOLITE.

THE theodolite is a brazen circular ring, divided into 360 degrees, &c. and having an index with sights, or a telescope, placed on the centre, about which the index is moveable; also a compass fixed to the centre, to point out courses and check the sights; the whole being fixed by the centre on a stand of a convenient height for use.

In using this instrument, an exact account, or field-book, of all measures and things necessary to be remarked in the plan, must be kept, from which to make out the plan on returning home from the ground.

Begin at such part of the ground, and measure in such directions as are judged most convenient; taking angles or directions to objects, and measuring such distances as appear

necessary, under the same restrictions as in the use of the plain table. And it is safest to fix the theodolite in the original position at every station, by means of fore and back objects, and the compass, exactly as in using the plain table; registering the number of degrees cut off by the index when directed to each object; and, at any station, placing the index at the same degree as when the direction towards that station was taken from the last preceding one, to fix the theodolite there in the original position.

The best method of laying down the aforesaid lines of direction, is to describe a pretty large circle; then quarter it, and lay on it the several numbers of degrees cut off by the index in each direction, and drawing lines from the centre to all these marked points in the circle. Then, by means of a parallel ruler, draw from station to station, lines parallel to the aforesaid lines drawn from the centre to the respective points in the circumference.

4. OF THE CROSS.

THE cross consists of two pair of sights set at right angles to each other, on a staff having a sharp point at the bottom, to fix in the ground.

The cross is very useful to measure small and crooked pieces of ground. The method is, to measure a base or chief line, usually in the longest direction of the piece, from corner to corner; and while measuring it, finding the places where perpendiculars would fall on this line, from the several corners and bends in the boundary of the piece, with the cross, by fixing it, by trials, on such parts of the line, as that through one pair of the sights both ends of the line may appear, and through the other pair the corresponding bends or corners; and then measuring the lengths of the said perpendiculars.

REMARKS.

Besides the fore-mentioned instruments, which are most commonly used, there are some others; as,

The perambulator, used for measuring roads, and other great distances, level ground, and by the sides of rivers. It has a wheel of $8\frac{1}{4}$ feet, or half a pole, in circumference, by the turning of which the machine goes forward: and the distance measured is pointed out by an index, which is moved round by clock-work.

Levels, with telescopic or other sights, are used to find the level between place and place, or how much one place is higher or lower than another. And in measuring any sloping or oblique line, either ascending or descending, a small

pocket level is useful for showing how many links for each chain are to be deducted, to reduce the line to the horizontal length.

An offset-staff is a very useful instrument, for measuring the offsets and other short distances. It is 10 links in length, being divided and marked at each of the 10 links.

Ten small arrows, or rods of iron or wood, are used to mark the end of every chain length, in measuring lines. And sometimes pickets, or staves with flags, are set up as marks or objects of direction.

Various scales are also used in protracting and measuring on the plan or paper; such as plane scales, line of chords, protractor, compasses, reducing scale, parallel and perpendicular rules, &c. Of plane scales, there should be several sizes, as a chain in 1 inch, a chain in $\frac{3}{4}$ of an inch, a chain in $\frac{1}{2}$ an inch, &c. And of these, the best for use are those that are laid on the very edges of the ivory scale, to mark off distances, without compasses.

SECTION II.

THE PRACTICE OF SURVEYING.

This part contains the several works proper to be done in the field, or the ways of measuring by all the instruments, and in all situations.

PROBLEM I.

To Measure a Line or Distance.

To measure a line on the ground with the chain: Having provided a chain, with 10 small arrows, or rods, to fix one into the ground, as a mark, at the end of every chain; two persons take hold of the chain, one at each end of it; and all the 10 arrows are taken by one of them, who goes foremost, and is called the leader; the other being called the follower, for distinction's sake.

A picket, or station-staff, being set up in the direction of the line to be measured, if there do not appear some marks naturally in that direction, they measure straight towards it, the leader fixing down an arrow at the end of every chain, which the follower always takes up, as he comes at it, till all the ten arrows are used. They are then all returned to the leader, to use over again. And thus the arrows are changed from the one to the other at every 10 chains length, till the whole line is finished; then the number of changes

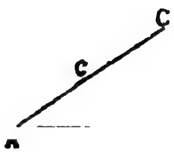
of the arrows shows the number of tens, to which the follower adds the arrows he holds in his hand, and the number of links of another chain over to the mark or end of the line. So, if there have been 3 changes of the arrows, and the follower hold 6 arrows, and the end of the line cut off 45 links more, the whole length of the line is set down in links thus, 3645.

When the ground is not level, but either ascending or descending; at every chain length, lay the offset-staff, or link-staff, down in the slope of the chain, on which lay the small pocket level, to show how many links or parts the slope line is longer than the true level one; then draw the chain forward so many links or parts, which reduces the line to the horizontal direction.

PROBLEM II.

To take Angles and Bearings.

Let B and C be two objects, or two pickets set up perpendicular; and let it be required to take their bearings, or the angles formed between them at any station A.



1. *With the Plain Table.*

The table being covered with a paper, and fixed on its stand; plant it at the station A, and fix a fine pin, or a foot of the compasses, in a proper point of the paper, to represent the place A: Close by the side of this pin lay the fiducial edge of the index, and turn it about, still touching the pin, till one object B can be seen through the sights: then by the fiducial edge of the index draw a line. In the same manner draw another line in the direction of the other object C. And it is done.

2. *With the Theodolite, &c.*

Direct the fixed sights along one of the lines, as AB, by turning the instrument about till the mark B is seen through these sights; and there screw the instrument fast. Then turn the moveable index round, till through its sights the other mark C is seen. Then the degrees cut by the index, on the graduated limb or ring of the instrument, show the quantity of the angle.

3. *With the Magnetic Needle and Compass.*

Turn the instrument or compass so, that the north end of the needle point to the flower-de-luce. Then direct the sights to one mark as B, and note the degrees cut by the needle. Next direct the sights to the other mark C, and note again the degrees cut by the needle. Then their sum or difference, as the case may be, will give the quantity of the angle BAC.

4. *By Measurement with the Chain, &c.*

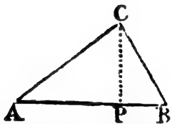
Measure one chain length, or any other length, along both directions, as to *b* and *c*. Then measure the distance *bc*, and it is done.—This is easily transferred to paper, by making a triangle A*bc* with these three lengths, and then measuring the angle A.

PROBLEM III.

To Survey a Triangular Field ABC.

1. *By the Chain.*

AP	794
AB	1321
PC	826



Having set up marks at the corners, which is to be done in all cases where there are not marks naturally; measure with the chain from A to P, where a perpendicular would fall from the angle C, and set up a mark at P, noting down the distance AP. Then complete the distance AB, by measuring from P to B. Having set down this measure, return to P, and measure the perpendicular PC. And thus, having the base and perpendicular, the area from them is easily found. Or having the place P of the perpendicular, the triangle is easily constructed.

Or, measure all the three sides with the chain, and note them down. From which the content is easily found, or the figure is constructed.

2. *By taking some of the Angles.*

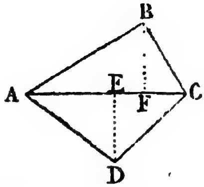
Measure two sides AB, AC, and the angle A between them. Or measure one side AB, and the two adjacent angles A and B. From either of these ways the figure is easily planned; then by measuring the perpendicular CP on the plan, and multiplying it by half AB, the content is found.

PROBLEM IV.

To Measure a Four-sided Field.

1. *By the Chain.*

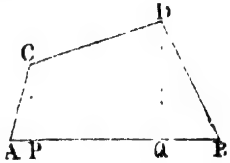
AE	214	210	DE
AF	362	306	BF
AC	592		



Measure along one of the diagonals, as AC; and either the two perpendiculars DE, BF, as in the last problem; or else the sides AB, BC, CD, DA. From either of which the figure may be planned and computed as before directed.

Otherwise, by the Chain.

AP	110	352	PC
AQ	745	595	QD
AB	1110		



Measure, on the longest side, the distances AP, AQ, AB; and the perpendiculars PC, QD.

2. *By taking some of the Angles.*

Measure the diagonal AC (see the last fig. but one), and the angles CAB, CAD, ACB, ACD.—Or measure the four sides, and any one of the angles, as BAD.

Thus.		Or thus.	
AC	591	AB	486
CAB	37° 20′	BC	394
CAD	41 15	CD	410
ACB	72 25	DA	462
ACD	54 40	BAD	78° 35′

PROBLEM V.

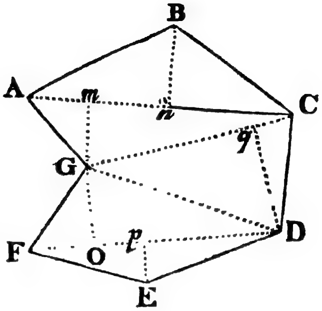
To Survey any Field by the Chain only.

HAVING set up marks at the corners, where necessary, of the proposed field ABCDEFG, walk over the ground, and consider how it can best be divided into triangles and trapeziums; and measure them separately, as in the last two problems. Thus, the following figure is divided into the two trapeziums ABCG, GDEF, and the triangle GCD. Then, in the first trapezium, beginning at A, measure the diagonal AC, and the

two perpendiculars Gm, Bn. Then the base GC, and the perpendicular Dq. Lastly, the diagonal DF, and the two perpendiculars pE, OG. All which measures write against the corresponding parts of a rough figure drawn to resemble the figure surveyed, or set them down in any other form you choose.

Thus.

Am	135	130	mG
An	410	180	nB
AC	550		
Cq	152	230	qD
CG	440		
FO	237	120	OG
Fp	288	80	pE
FD	520		



Or thus.

Measure all the sides AB, BC, CD, DE, EF, FG, GA; and the diagonals AC, CG, GD, DF.

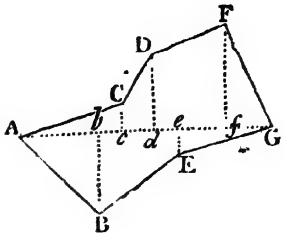
Otherwise.

Many pieces of land may be very well surveyed, by measuring any base line, either within or without them, with the perpendiculars let fall on it from every corner. For they are by those means divided into several triangles and trapezoids, all whose parallel sides are perpendicular to the base line; and the sum of these triangles and trapeziums will be equal to the figure proposed if the base line fall within it; if not, the sum of the parts which are without being taken from the sum of the whole which are both within and without, will leave the area of the figure proposed.

In pieces that are not very large, it will be sufficiently exact to find the points, in the base line, where the several perpendiculars will fall, by means of the *cross*, or even by judging by the eye only, and from thence measuring to the corners for the lengths of the perpendiculars.—And it will be most convenient to draw the line so as that all the perpendiculars may fall within the figure.

Thus, in the following figure, beginning at A, and measuring along the line AG, the distances and perpendiculars on the right and left are as below.

A*b*	315	350 *b*B
A*c*	440	70 *c*C
A*d*	585	320 *d*D
A*e*	610	50 *e*E
A*f*	990	470 *f*F
AG	1020	0

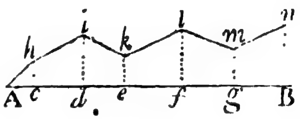


PROBLEM VI.

To Measure the Offsets.

A*hiklmn* being a crooked hedge, or brook, &c. From A measure in a straight direction along the side of it to B. And in measuring along this line AB, observe when you are directly opposite any bends or corners of the boundary, as at *c*, *d*, *e*, &c.; and from these measure the perpendicular offsets *ch*, *di*, &c. with the offset-staff, if they are not very large, otherwise with the chain itself; and the work is done. The register, or field-book, may be as follows:

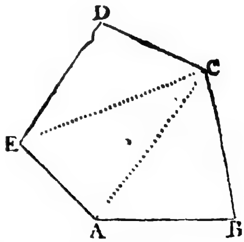
	Offs. left.	Base line AB	
	0	⊙	A
ch	62	45	A*c*
di	84	220	A*d*
ek	70	340	A*e*
fl	98	510	A*f*
gm	57	634	A*g*
B*n*	91	785	AB



PROBLEM VII.

To Survey any Field with the Plain Table.

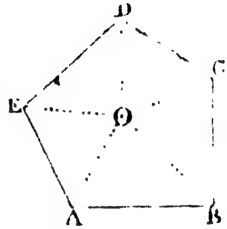
1. *From one Station.*



PLANT the table at any angle as C, from which all the other angles, or marks set up, can be seen; turn the table about till the needle point to the flower-de-luce; and there screw it fast. Make a point for C on the paper on the table, and lay the edge of the index to C, turning it about C till through the sights you see the mark D: and by the edge of the index draw a dry or obscure line: then measure the distance CD, and lay that distance down on the line CD. Then turn the index about the point C, till the mark E be seen through the sights,

by which draw a line, and measure the distance to E, laying it on the line from C to E. In like manner determine the positions of CA and CB, by turning the sights successively to A and B; and lay the lengths of those lines down. Then connect the points, by drawing the black lines CD, DE, EA, AB, BC, for the boundaries of the field.

2. From a Station within the Field.



When all the other parts cannot be seen from one angle, choose some place O within, or even without, if more convenient, from which the other parts can be seen. Plant the table at O, then fix it with the needle north, and mark the point O on it. Apply the index successively to O, turning it round with the sights to each angle, A, B, C, D, E, drawing dry lines to them by the edge of the index; then measuring the distances OA, OB, &c. and laying them down on those lines. Lastly, draw the boundaries AB, BC, CD, DE, EA.

3. By going Round the Figure.

When the figure is a wood, or water, or when from some other obstruction you cannot measure lines across it; begin at any point A, and measure around it, either within or without the figure, and draw the directions of all the sides, thus: Plant the table at A; turn it with the needle to the north or flower-de-luce; fix it, and mark the point A. Apply the index to A, turning it till you can see the point E, and there draw a line: then the point B, and there draw a line: then measure these lines, and lay them down from A to E and B. Next move the table to B, lay the index along the line AB, and turn the table about till you can see the mark A, and screw fast the table; in which position also the needle will again point to the flower-de-luce, as it will do indeed at every station when the table is in the right position. Here turn the index about B till through the sights you see the mark C; there draw a line, measure BC, and lay the distance on that line after you have set down the table at C. Turn it then again into its proper position, and in like manner find the next line CD. And so on quite around by E, to A again. Then the proof of the work will be the joining at A: for if the work be all right, the last direction EA on the ground, will pass exactly through the point A on the paper; and the measured distance will also reach exactly to A. If these do not coincide, or nearly so, some error has been committed, and the work must be examined over again.

PROBLEM VIII.

To Survey a Field with the Theodolite, &c.

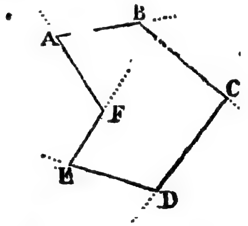
1. *From One Point or Station.*

When all the angles can be seen from one point, as the angle C (first fig. to last prob.) place the instrument at C, and turn it about, till through the fixed sights you see the mark B, and there fix it. Then turn the moveable index about till the mark A be seen through the sights, and note the degrees cut on the instrument. Next turn the index successively to E and D, noting the degrees cut off at each; which gives all the angles BCA, BCE, BCD. Lastly, measure the lines CB, CA, CE, CD; and enter the measures in a field-book, or rather against the corresponding parts of a rough figure drawn by guess to resemble the field.

2. *From a Point Within or Without.*

Plant the instrument at O (last fig.), and turn it about till the fixed sights point to any object, as A; and there screw it fast. Then turn the moveable index round till the sights point successively to the other points E, D, C, B, noting the degrees cut off at each of them; which gives all the angles round the point O. Lastly, measure the distances OA, OB, OC, OD, OE, noting them down as before, and the work is done.

3. *By going Round the Field.*



By measuring round, either within or without the field, proceed thus. Having set up marks at B, C, &c. near the corners as usual, plant the instrument at any point A, and turn it till the fixed index be in the direction AB, and there screw it fast: then turn the moveable index to the direction AF; and the degrees cut off will be the angle A. Measure the line AB, and plant the instrument at B, and there in the same manner observe the angle A. Then measure BC, and observe the angle C. Then measure the distance CD, and take the angle D. Then measure DE, and take the angle E. Then measure EF, and take the angle F. And lastly measure the distance FA.

To prove the work; add all the inward angles, A, B, C, &c. together; for when the work is right, their sum will be equal to twice as many right angles as the figure has sides, wanting 4 right angles. But when there is an angle, as F, that bends inwards, and you measure the external angle,

which is less than two right angles, subtract it from 4 right angles, or 360 degrees, to give the internal angle greater than a semicircle or 180 degrees.

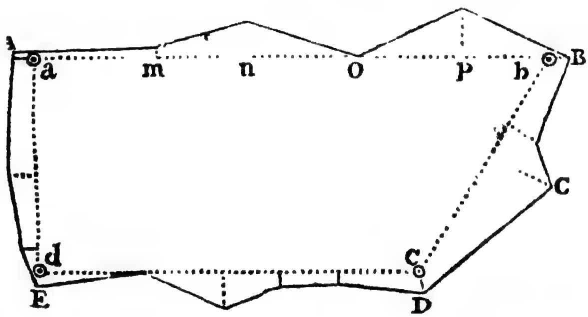
Otherwise.

Instead of observing the internal angles, we may take the external angles, formed without the figure by producing the sides farther out. And in this case, when the work is right, their sum altogether will be equal to 360 degrees. But when one of them, as F, runs inwards, subtract it from the sum of the rest, to leave 360 degrees.

PROBLEM IX.

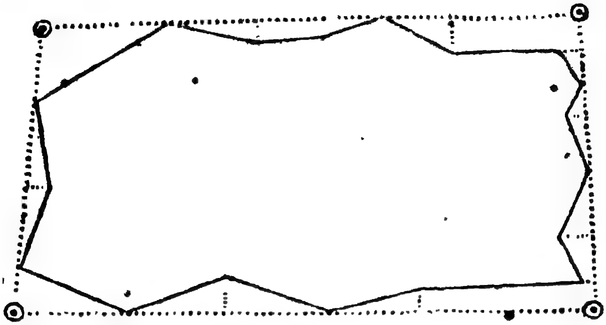
To Survey a Field with Crooked Hedges, &c.

WITH any of the instruments, measure the lengths and positions of imaginary lines running as near the sides of the field as you can; and, in going along them, measure the offsets in the manner before taught; then you will have the plan on the paper in using the plain table, drawing the crooked hedges through the ends of the offsets; but in surveying with the theodolite, or other instrument, set down the measures properly in a field-book, or memorandum-book, and plan them after returning from the field, by laying down all the lines and angles.



So, in surveying the piece ABCDE, set up marks, *a*, *b*, *c*, *d*, dividing it so as to have as few sides as may be. Then begin at any station, *a*, and measure the lines *ab*, *bc*, *cd*, *da*, taking their positions, or the angles *a*, *b*, *c*, *d*; and, in going along the lines, measure all the offsets, as at *m*, *n*, *o*, *p*, &c. along every station-line.

And this is done either within the field, or without, as may be most convenient. When there are obstructions within, as wood, water, hills, &c. then measure without, as in the next following figure.



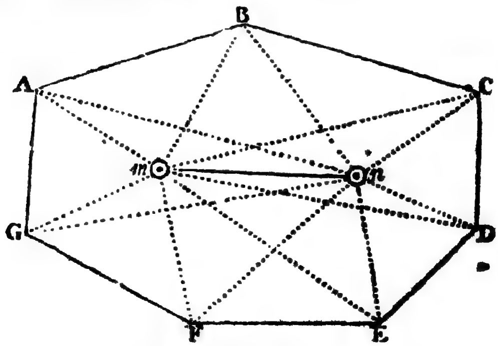
PROBLEM X.

To Survey a Field, or any other Thing, by Two Stations.

This is performed by choosing two stations from which all the marks and objects can be seen; then measuring the distance between the stations, and at each station taking the angles formed by every object from the station line or distance.

The two stations may be taken either within the bounds, or in one of the sides, or in the direction of two of the objects, or quite at a distance and without the bounds of the objects or part to be surveyed.

In this manner, not only grounds may be surveyed, without even entering them, but a map may be taken of the principal parts of a county, or the chief places of a town, or any part of a river or coast surveyed, or any other inaccessible objects; by taking two stations, on two towers, or two hills, or such-like.



PROBLEM XI.

To Survey a Large Estate.

If the estate be very large, and contain a great number of fields, it cannot well be done by surveying all the fields

singly, and then putting them together; nor can it be done by taking all the angles and boundaries that enclose it. For in these cases, any small errors will be so much increased, as to render it very much distorted. But proceed as below.

1. Walk over the estate two or three times, in order to get a perfect idea of it, or till you can keep the figure of it pretty well in mind. And to help your memory, draw an eye-draught of it on paper, or at least of the principal parts of it, to guide you; setting the names within the fields in that draught.

2. Choose two or more eminent places in the estate, for stations, from which all the principal parts of it can be seen: selecting these stations as far distant from one another as convenient.

3. Take such angles, between the stations, as you think necessary, and measure the distances from station to station, always in a right line: these things must be done, till you get as many angles and lines as are sufficient for determining all the points of station. And in measuring any of these station-distances, mark accurately where these lines meet with any hedges, ditches, roads, lanes, paths, rivulets, &c.; and where any remarkable object is placed, by measuring its distance from the station-line; and where a perpendicular from it cuts that line. And thus as you go along any main station-line, take offsets to the ends of all hedges, and to any pond, house, mill, bridge, &c. noting every thing down that is remarkable.

4. As to the inner parts of the estate, they must be determined, in like manner, by new station-lines: for, after the main stations are determined, and every thing adjoining to them, then the estate must be subdivided into two or three parts by new station-lines; taking inner stations at proper places, where you can have the best view. Measure these station-lines as you did the first, and all their intersections with hedges, and offsets to such objects as appear. Then proceed to survey the adjoining fields, by taking the angles that the sides make with the station-line, at the intersections; and measuring the distances to each corner, from the intersections. For the station-lines will be the bases to all the future operations; the situation of all parts being entirely dependent on them; and therefore they should be taken of as great length as possible; and it is best for them to run along some of the hedges or boundaries of one or more fields, or to pass through some of their angles. All things being determined for these stations, you must take more inner stations, and continue to divide and subdivide till at last you

come to single fields: repeating the same work for the inner stations as for the outer ones, till all is done; and close the work as often as you can, and in as few lines as possible.

5. An estate may be so situated that the whole cannot be surveyed together; because one part of the estate cannot be seen from another. In this case, you may divide it into three or four parts, and survey the parts separately, as if they were lands belonging to different persons; and at last join them together.

6. As it is necessary to protract or lay down the work as you proceed in it, you must have a scale of a due length to do it by. To get such a scale, measure the whole length of the estate in chains; then consider how many inches long the map is to be; and from these will be known how many chains you must have in an inch; then make the scale accordingly, or choose one already made.

PROBLEM XII.

To Survey a County, or large Tract of Land.

1. CHOOSE two, three, or four eminent places, for stations; such as the tops of high hills or mountains, towers, or church steeples, which may be seen from one another; from which most of the towns and other places of note may also be seen; and so as to be as far distant from one another as possible. On these places raise beacons, or long poles, with flags of different colours flying at them, so as to be visible from all the other stations.

2. At all the places which you would set down in the map, plant long poles, with flags at them of several colours, to distinguish the places from one another; fixing them on the tops of church steeples, or the tops of houses; or in the centres of smaller towns and villages.

These marks then being set up at a convenient number of places, and such as may be seen from both stations; go to one of these stations, and, with an instrument to take angles, standing at that station, take all the angles between the other station and each of these marks. Then go to the other station, and take all the angles between the first station and each of the former marks, setting them down with the others, each against its fellow with the same colour. You may, if convenient, also take the angles at some third station, which may serve to prove the work, if the three lines intersect in that point where any mark stands. The marks must stand till the observations are finished at both stations; and

then they may be taken down, and set up at new places. The same operations must be performed at both stations, for these new places; and the like for others. The instrument for taking angles must be an exceeding good one, made on purpose with telescopic sights, and of a good length of radius.

3. And though it be not absolutely necessary to measure any distance, because, a stationary line being laid down from any scale, all the other lines will be proportional to it; yet it is better to measure some of the lines, to ascertain the distances of places in miles, and to know how many geometrical miles there are in any length; as also from thence to make a scale to measure any distance in miles. In measuring any distance, it will not be exact enough to go along the high roads; which, by reason of their turnings and windings, hardly ever lie in a right line between the stations; which must cause endless reductions, and require great trouble to make it a right line; for which reason it can never be exact. But a better way is to measure in a straight line with a chain, between station and station, over hills and dales, or level fields and all obstacles. Only in case of water, woods, towns, rocks, banks, &c. where we cannot pass, such parts of the line must be measured by the methods of inaccessible distances; and besides, allowing for ascents and descents, when they are met with. A good compass, that shows the bearing of the two stations, will always direct us to go straight, when the two stations cannot be seen; and in the progress, if we can go straight, offsets may be taken to any remarkable places, likewise noting the intersection of the station-line with all roads, rivers, &c.

4. From all the stations, and in the whole progress, we must be very particular in observing sea-coasts, river-mouths, towns, castles, houses, churches, mills, trees, rocks, sands, roads, bridges, fords, ferries, woods, hills, mountains, rills, brooks, parks, beacons, sluices, floodgates, locks, &c., and in general every thing that is remarkable.

5. After we have done with the first and main station-lines, which command the whole county; we must then take inner stations, at some places already determined; which will divide the whole into several partitions: and from these stations we must determine the places of as many of the remaining towns as we can. And if any remain in that part, we must take more stations, at some places already determined; from which we may determine the rest. And thus go through all the parts of the county, taking station after station, till we have determined the whole. And in

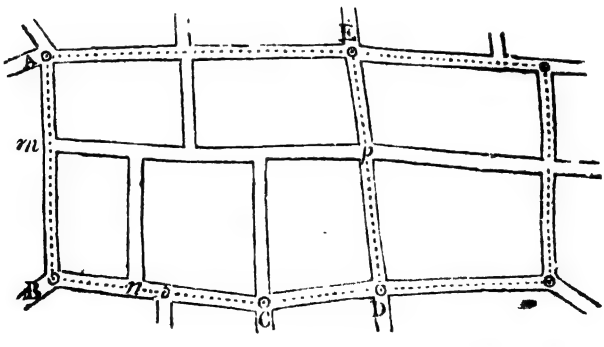
general the station distances must always pass through such remarkable points as have been determined before, by the former stations.

PROBLEM XIII.

To Survey a Town or City.

This may be done with any of the instruments for taking angles, but best of all with the plain table, where every minute part is drawn while in sight. Instead of the common surveying or Gunter's chain, it will be best, for this purpose, to have a chain 50 feet long, divided into 50 links of one foot each, and an offset-staff of 10 feet long.

Begin at the meeting of two or more of the principal streets, through which you can have the longest prospects, to get the longest station-lines: there having fixed the instrument, draw lines of direction along those streets, using two men as marks, or poles set in wooden pedestals, or perhaps some remarkable places in the houses at the farther ends, as windows, doors, corners, &c. Measure these lines with the chain, taking offsets with the staff, at all corners of streets, bendings, or windings, and to all remarkable things, as churches, markets, halls, colleges, eminent houses, &c. Then remove the instrument to another station, along one of these lines; and there repeat the same process as before. And so on till the whole is finished.



Thus, fix the instrument at A, and draw lines in the direction of all the streets meeting there; then measure AB, noting the street on the left at *m*. At the second station B, draw the directions of the streets meeting there; and measure from B to C, noting the places of the streets at *n* and *o*

as you pass by them. At the third station C, take the direction of all the streets meeting there, and measure CD. At D do the same, and measure DE, noting the place of the cross streets at *p*. And in this manner go through all the principal streets. This done, proceed to the smaller and intermediate streets; and lastly to the lanes, alleys, courts, yards, and every part that it may be thought proper to represent in the plan.

PROBLEM XIV.

To lay down the Plan of any Survey.

If the survey was taken with the plain table, we have a rough plan of it already on the paper which covered the table. But if the survey was with any other instrument, a plan of it is to be drawn from the measures that were taken in the survey; and first of all a rough plan on paper.

To do this, you must have a set of proper instruments, for laying down both lines and angles, &c.; as scales of various sizes, the more of them, and the more accurate, the better, scales of chords, protractors, perpendicular and parallel rulers, &c. Diagonal scales are best for the lines, because they extend to three figures, or chains, and links, which are 100 parts of chains. But in using the diagonal scale, a pair of compasses must be employed, to take off the lengths of the principal lines very accurately. But a scale with a thin edge divided, is much readier for laying down the perpendicular offsets to crooked hedges, and for marking the places of those offsets on the station-line; which is done at only one application of the edge of the scale to that line, and then pricking off all at once the distances along it. Angles are to be laid down, either with a good scale of chords, which is perhaps the most accurate way, or with a large protractor, which is much readier when many angles are to be laid down at one point, as they are pricked off all at once round the edge of the protractor.

In general, all lines and angles must be laid down on the plan in the same order in which they were measured in the field, and in which they are written in the field-book; laying down first the angles for the position of lines, next the lengths of the lines, with the places of the offsets, and then the lengths of the offsets themselves, all with dry or obscure lines; then a black line drawn through the extremities of all the offsets, will be the edge or bounding line of the field, &c. After the principal bounds and lines are laid down,

and made to fit or close properly, proceed next to the smaller objects, till you have entered every thing that ought to appear in the plan, as houses, brooks, trees, hills, gates, stiles, roads, lanes, mills, bridges, woodlands, &c. &c.

The north side of a map or plan is commonly placed uppermost, and a meridian is somewhere drawn, with the compass or flower-de-luce pointing north. Also, in a vacant part, a scale of equal parts or chains is drawn, with the title of the map in conspicuous characters, and embellished with a compartment. Hills are shadowed, to distinguish them in the map. Colour the hedges with different colours; represent hilly grounds by broken hills and valleys; draw single dotted lines for foot-paths, and double ones for horse or carriage roads. Write the name of each field and remarkable place within it, and, if you choose, its contents in acres, roods, and perches.

In a very large estate, or a county, draw vertical and horizontal lines through the map, denoting the spaces between them by letters placed at the top, and bottom, and sides, for readily finding any field or other object mentioned in a table.

In mapping counties, and estates that have uneven grounds of hills and valleys, reduce all oblique lines, measured uphill and down-hill, to horizonal straight lines, if that was not done during the survey, before they were entered in the field-book, by making a proper allowance to shorten them. For which purpose there is commonly a small table engraven on some of the instruments for surveying.

THE NEW METHOD OF SURVEYING.

PROBLEM XV.

To Survey and Plan by the New Method.

In the former method of measuring a large estate, the accuracy of it depends both on the correctness of the instruments, and on the care in taking the angles. To avoid the errors incident to such a multitude of angles, other methods have of late years been used by some few skilful surveyors: the most practical, expeditious, and correct, seems to be the following, which is performed, without taking angles, by measuring with the chain only.

Choose two or more eminences, as grand stations, and measure a principal base line from one station to another;

noting every hedge, brook, or other remarkable object, as you pass by it; measuring also such short perpendicular lines to the bends of hedges as may be near at hand. From the extremities of this base line, or from any convenient parts of the same, go off with other lines to some remarkable object situated towards the sides of the estate, without regarding the angles they make with the base line or with one another; still remembering to note every hedge, brook, or other object, that you pass by. These lines, when laid down by intersections, will, with the base line, form a grand triangle on the estate; several of which, if need be, being thus measured and laid down, you may proceed to form other smaller triangles and trapezoids on the sides of the former: and so on till you finish with the enclosures individually. By which means a kind of skeleton of the estate may first be obtained, and the chief lines serve as the bases of such triangles and trapezoids as are necessary to fill up all the interior parts.

The field-book is ruled into three columns, as usual. In the middle one are set down the distances on the chain-line, at which any mark, offset, or other observation, is made; and in the right and left hand columns are entered the offsets and observations made on the right and left hand respectively of the chain-line; sketching on the sides the shape or resemblance of the fences or boundaries.

It is of great advantage, both for brevity and perspicuity, to begin at the bottom of the leaf, and write upwards; denoting the crossing of fences, by lines drawn across the middle column, or only a part of such a line on the right and left opposite the figures, to avoid confusion; and the corners of fields, and other remarkable turns in the fences where offsets are taken to, by lines joining in the manner the fences do; as will be best seen by comparing the book with the plan annexed to the field-book following, p. 76.

The letter in the left-hand corner at the beginning of every line, is the mark or place measured *from;* and that at the right-hand corner at the end, is the mark measured *to:* but when it is not convenient to go exactly from a mark, the place measured from is described *such a distance* from *one mark* towards *another;* and where a former mark is not measured to, the exact place is ascertained by saying, turn to the right or left hand, *such a distance to such a mark*, it being always understood that those distances are taken in the chain-line.

The characters used are, ⌈ for *turn to the right hand,* ⌉ for *turn to the left hand,* and ⌃ placed over an offset, to show that it is not taken at right angles with the chain-

line, but in the direction of some straight fence; being chiefly used when crossing their directions; which is a better way of obtaining their true places than by offsets at right angles.

When a line is measured whose position is determined, either by former work (as in the case of producing a given line, or measuring from one known place or mark to another) or by itself (as in the third side of the triangle), it is called a *fast line*, and a double line across the book is drawn at the conclusion of it; but if its position is not determined (as in the second side of the triangle), it is called a *loose line*, and a single line is drawn across the book. When a line becomes determined in position, and is afterwards continued farther, a double line half through the book is drawn.

When a loose line is measured, it becomes absolutely necessary to measure some other line that will determine its position. Thus, the first line *ah* or *bh*, being the base of a triangle, is always determined; but the position of the second side *hj* does not become determined, till the third side *jb* is measured; then the position of both is determined, and the triangle may be constructed.

At the beginning of a line, to fix a loose line to the mark or place measured from, the sign of turning to the right or left hand must be added, as at *h* in the second, and *j* in the third line; otherwise a stranger, when laying down the work, may as easily construct the triangle *hjb* on the wrong side of the line *ah*, as on the right one: but this error cannot be fallen into, if the sign above named be carefully observed.

In choosing a line to fix a loose one, care must be taken that it does not make a very acute or obtuse angle; as in the triangle *p*B*r*, by the angle at B being very obtuse, a small deviation from truth, even the breadth of a point at *p* or *r*, would make the error at B, when constructed, very considerable; but by constructing the triangle *p*B*q*, such a deviation is of no consequence.

Where the words *leave off* are written in the field-book, it signifies that the taking of offsets is from thence discontinued; and of course something is wanting between that and the next offset, to be afterwards determined by measuring some other line.

The field-book for this method, and the plan drawn from it, are contained in the four following pages, engraven on copper plates; answerable to which the pupil is to draw a plan from the measures in the field-book, of a larger size, viz. to a scale of a double size will be convenient, such a

scale being also found on most instruments. In doing this, begin at the commencement of the field-book, or bottom of the first page, and draw the first line *ah* in any direction at pleasure, and then the next two sides of the first triangle *bhj* by sweeping intersected arcs; and so all the triangles in the same manner, after each other in their order; and afterwards setting the perpendicular and other offsets at their proper places, and through the ends of them drawing the bounding fences.

Note. That the field-book begins at the bottom of the first page, and reads up to the top; hence it goes to the bottom of the next page, and to the top; and thence it passes from the bottom of the third page to the top, which is the end of the field-book. The several marks measured to or from, are here denoted by the letters of the alphabet, first the small ones, *a*, *b*, *c*, *d*, &c, and after them the capitals *A*, *B*, *C*, *D*, &c. But, instead of these letters, some surveyors use the numbers in order, 1, 2, 3, 4, &c.

OF THE OLD KIND OF FIELD-BOOK.

In surveying with the plain table, a field-book is not used, as every thing is drawn on the table immediately when it is measured. But in surveying with the theodolite, or any other instrument, some kind of a field-book must be used, to write down in it a register or account of all that is done and occurs relative to the survey in hand.

This book every one contrives and rules as he thinks fittest for himself. The following is a specimen of a form which has been formerly used. It is ruled in three columns, as in the next page.

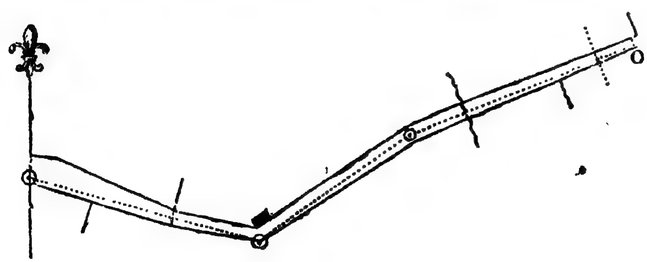
Here ⊙ 1 is the first station, where the angle or bearing is $105^{\circ}\ 25'$. On the left, at 73 links in the distance or principal line, is an offset of 92; and at 610 an offset of 24 to a cross hedge. On the right, at 0, or the beginning, an offset 25 to the corner of the field; at 248 Brown's boundary hedge commences; at 610 an offset 35; and at 954, the end of the first line, the 0 denotes its terminating in the hedge. And so on for the other stations.

A line is drawn under the work, at the end of every station line, to prevent confusion.

Form of this Field-book.

Offsets and Remarks on the left.	Stations, Bearings, and Distances.	Offsets and Remarks on the right.
	⊙ 1	
	105° 25′	
80	00	25 corner
92	73	
	248	Brown's hedge
a cross hedge 21	610	35
	954	00
	⊙ 2	
	53° 10′	
house corner 51	25	21
	120	29 a tree
34	734	40 a stile
	⊙ 3	
	67° 20′	
	61	35
a brook 30	248	
	639	16 a spring
foot path 16	810	
cross hedge 18	973	20 a pond

Then the plan, on a small scale drawn from the above field-book, will be as in the following figure. But the pupil may draw a plan of 3 or 4 times the size on his paper book. The dotted lines denote the 3 chain or measured lines, and the black lines the boundaries on the right and left.



But some skilful surveyors now make use of a different method for the field-book, namely, beginning at the bottom of the page and writing upwards; sketching also a neat

boundary on either hand, resembling the parts near the measured lines as they pass along; an example of which was given in the new method of surveying, in the preceding page.

In smaller surveys and measurements, a good way of setting down the work, is to draw by the eye, on a piece of paper, a figure resembling that which is to be measured; and so writing the dimensions, as they are found, against the corresponding parts of the figure. And this method may be practised to a considerable extent, even in the larger surveys.

SECTION III.

OF COMPUTING AND DIVIDING.

PROBLEM XVI.

To Compute the Contents of Fields.

1. Compute the contents of the figures as divided into triangles, or trapeziums, by the proper rules for these figures laid down in measuring; multiplying the perpendiculars by the diagonals or bases, both in links, and divide by 2; the quotient is acres, after having cut off five figures on the right for decimals. Then bring these decimals to roods and perches, by multiplying first by 4, and then by 40. An example of which is given in the description of the chain, pag. 53.

2. In small and separate pieces, it is usual to compute their contents from the measures of the lines taken in surveying them, without making a correct plan of them.

3. In pieces bounded by very crooked and winding hedges, measured by offsets, all the parts between the offsets are most accurately measured separately as small trapezoids.

4. Sometimes such pieces as that last mentioned are computed by finding a mean breadth, by adding all the offsets together, and dividing the sum by the number of them, accounting that for one of them where the boundary meets the station-line (which increases the number of them by 1, for the divisor, though it does not increase the sum or quantity to be divided); then multiply the length by that mean breadth.

5. But in larger pieces and whole estates, consisting of

many fields, it is the common practice to make a rough plan of the whole, and from it compute the contents, quite independent of the measures of the lines and angles that were taken in surveying. For then new lines are drawn in the fields on the plans, so as to divide them into trapeziums and triangles, the bases and perpendiculars of which are measured on the plan by means of the scale from which it was drawn, and so multiplied together for the contents. In this way, the work is very expeditiously done, and sufficiently correct; for such dimensions are taken as afford the most easy method of calculation; and among a number of parts, thus taken and applied to a scale, though it be likely that some of the parts will be taken a small matter too little, and others too great, yet they will, on the whole, in all probability, very nearly balance one another, and give a sufficiently accurate result. After all the fields and particular parts are thus computed separately, and added all together into one sum; calculate the whole estate independent of the fields, by dividing it into large and arbitrary triangles and trapeziums, and add these also together. Then if this sum be equal to the former, or nearly so, the work is right; but if the sums have any considerable difference, it is wrong, and they must be examined, and re-computed, till they nearly agree.

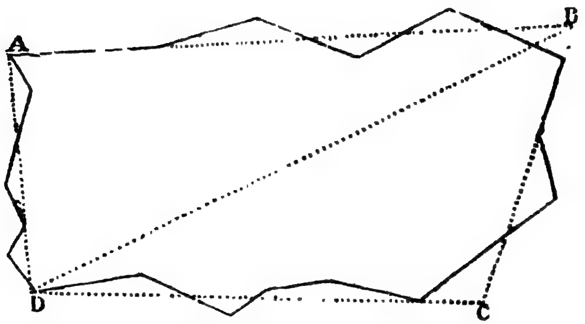
6. But the chief art in computing, consists in finding the contents of pieces bounded by curved or very irregular lines, or in reducing such crooked sides of fields or boundaries to straight lines, that shall enclose the same or equal area with those crooked sides, and so obtain the area of the curved figure by means of the right-lined one, which will commonly be a trapezium. Now this reducing the crooked sides to straight ones, is very easy and accurately performed in this manner:—Apply the straight edge of a thin, clear piece of lantern-horn to the crooked line, which is to be reduced, in such a manner, that the small parts cut off from the crooked figure by it, may be equal to those which are taken in: which equality of the parts included and excluded you will presently be able to judge of very nicely by a little practice: then with a pencil, or point of a tracer, draw a line by the straight edge of the horn. Do the same by the other sides of the field or figure. So shall you have a straight-sided figure equal to the curved one; the content of which, being computed as before directed, will be the content of the crooked figure proposed.

Or, instead of the straight edge of the horn, a horse-hair, or fine thread, may be applied across the crooked sides in the same manner; and the easiest way of using the thread, is

to string a small slender bow with it, either of wire, or cane, or whale-bone, or such-like slender elastic matter; for the bow keeping it always stretched, it can be easily and neatly applied with one hand, while the other is at liberty to make two marks by the side of it, to draw the straight line by.

EXAMPLE.

Thus, let it be required to find the contents of the same figure as in Prob. IX, page 66, to a scale of 4 chains to an inch.



Draw the 4 dotted straight lines AB, BC, CD, DA, cutting off equal quantities on both sides of them, which they do as near as the eye can judge: so is the crooked figure reduced to an equivalent right-lined one of 4 sides, ABCD. Then draw the diagonal BD, which, by applying a proper scale to it, measures suppose 1256. Also the perpendicular, or nearest distance from A to this diagonal, measures 456; and the distance of C from it, is 428.

Then, half the sum of 456 and 428, multiplied by the diagonal 1256, gives 555152 square links, or 5 acres, 2 roods, 8 perches, the content of the trapezium, or of the irregular crooked piece.

As a general example of this practice, let the contents be computed of all the fields separately in the foregoing plan facing page 77, and, by adding the contents altogether, the whole sum or content of the estate will be found nearly equal to $103\frac{1}{2}$ acres. Then, to prove the work, divide the whole plan into two parts, by a pencil line drawn across it any way near the middle, as from the corner *l* on the right, to the corner near *s* on the left; then, by computing these two large parts separately, their sum must be nearly equal to the former sum, when the work is all right.

PROBLEM XVII.

To Transfer a Plan to Another Paper, &c.

AFTER the rough plan is completed, and a fair one is wanted; this may be done by any of the following methods.

First Method.—Lay the rough plan on the clean paper, keeping them always pressed flat and close together, by weights laid on them. Then with the point of a fine pin or pricker, prick through all the corners of the plan to be copied. Take them asunder, and connect the pricked points on the clean paper, with lines; and it is done. This method is only to be practised in plans of such figures as are small and tolerably regular, or bounded by right lines.

Second Method.—Rub the back of the rough plan over with black-lead powder; and lay this blacked part on the clean paper on which the plan is to be copied, and in the proper position. Then, with the blunt point of some hard substance, as brass, or such-like, trace over the lines of the whole plan; pressing the tracer so much, as that the black lead under the lines may be transferred to the clean paper: after which, take off the rough plan, and trace over the leaden marks with common ink, or with Indian ink.—Or, instead of blacking the rough plan, we may keep constantly a blacked paper to lay between the plans.

Third Method.—Another method of copying plans, is by means of squares. This is performed by dividing both ends and sides of the plan which is to be copied into any convenient number of equal parts, and connecting the corresponding points of division with lines: which will divide the plan into a number of small squares. Then divide the paper, on which the plan is to be copied, into the same number of squares, each equal to the former when the plan is to be copied of the same size, but greater or less than the others, in the proportion in which the plan is to be increased or diminished, when of a different size. Lastly, copy into the clean squares the parts contained in the corresponding squares of the old plan; and you will have the copy, either of the same size, or greater or less in any proportion.

Fourth Method.—A fourth method is by the instrument called a pentagraph, which also copies the plan in any size required: for this purpose, also, Professor Wallace's eidograph may be advantageously employed.

Fifth Method.—A very neat method, at least in copying

from a fair plan, is this. Procure a copying frame or glass, made in this manner; namely, a large square of the best window glass, set in a broad frame of wood, which can be raised up to any angle, when the lower side of it rests on a table. Set this frame up to any angle before you, facing a strong light; fix the old plan and clean paper together, with several pins quite around, to keep them together, the clean paper being laid uppermost, and over the face of the plan to be copied. Lay them, with the back of the old plan, on the glass; namely, that part which you intend to begin at to copy first; and by means of the light shining through the papers, you will very distinctly perceive every line of the plan through the clean paper. In this state then trace all the lines on the paper with a pencil. Having drawn that part which covers the glass, slide another part over the glass, and copy it in the same manner. Then another part: and so on, till the whole is copied. Then take them asunder, and trace all the pencil lines over with a fine pen and Indian ink, or with common ink. And thus you may copy the finest plan, without injuring it in the least.

OF ARTIFICERS' WORKS,

AND

TIMBER MEASURING.

I. OF THE CARPENTER'S OR SLIDING RULE.

The Carpenter's or Sliding Rule, is an instrument much used in measuring of timber and artificers' works, both for taking the dimensions, and computing the contents.

The instrument consists of two equal pieces, each a foot in length, which are connected together by a folding joint.

One side or face of the rule is divided into inches, and eighths, or half-quarters. On the same face also are several plane scales, divided into twelfth parts by diagonal lines; which are used in planning dimensions that are taken in feet and inches. The edge of the rule is commonly divided decimally, or into tenths; namely, each foot into ten equal

parts, and each of these into ten parts again: so that by means of this last scale, dimensions are taken in feet, tenths, and hundredths, and multiplied as common decimal numbers, which is the best way.

On the one part of the other face are four lines, marked A, B, C, D; the two middle ones B and C being on a slider, which runs in a groove made in the stock. The same numbers serve for both these two middle lines, the one being above the numbers, and the other below.

These four lines are logarithmic ones, and the three A, B, C, which are all equal to one another, are double lines, as they proceed twice over from 1 to 10. The other or lowest line, D, is a single one, proceeding from 4 to 40. It is also called the girt line, from its use in computing the contents of trees and timber; and on it are marked WG at 17·15, and AG at 18·95, the wine and ale gage points, to make this instrument serve the purpose of a gaging rule.

On the other part of this face, there is a table of the value of a load, or 50 cubic feet of timber, at all prices, from 6 pence to 2 shillings a foot.

When 1 at the beginning of any line is accounted 1, then the 1 in the middle will be 10, and the 10 at the end 100; but when 1 at the beginning is counted 10, then the 1 in the middle is 100, and the 10 at the end 1000; and so on. And all the smaller divisions are altered proportionally.

II. ARTIFICERS' WORK.

ARTIFICERS compute the contents of their works by several different measures. As,

Glazing and masonry, by the foot; Painting, plastering, paving, &c. by the yard, of 9 square feet: Flooring, partitioning, roofing, tiling, &c. by the square of 100 square feet:

And brickwork, either by the yard of 9 square feet, or by the perch, or square rod or pole, containing $272\frac{1}{4}$ square feet, or $30\frac{1}{4}$ square yards, being the square of the rod or pole of $16\frac{1}{2}$ feet or $5\frac{1}{2}$ yards long.

As this number $272\frac{1}{4}$ is troublesome to divide by, the $\frac{1}{4}$ is often omitted in practice, and the content in feet divided only by the 272.

All works, whether superficial or solid, are computed by the rules proper to the figure of them, whether it be a triangle, or rectangle, a parallelopiped, or any other figure.

III. BRICKLAYERS' WORK.

Brickwork is estimated at the rate of a brick and a half thick. So that if a wall be more or less than this standard thickness, it must be reduced to it, as follows:

Multiply the superficial content of the wall by the number of half bricks in the thickness, and divide the product by 3.

The dimensions of a building may be taken by measuring half round on the outside and half round on the inside; the sum of these two gives the compass of the wall, to be multiplied by the height, for the content of the materials.

Chimneys are commonly measured as if they were solid, deducting only the vacuity from the hearth to the mantle, on account of the trouble of them. All windows, doors, &c. are to be deducted out of the contents of the walls in which they are placed.

The dimensions of a common bare brick are, $8\frac{1}{2}$ inches long, 4 inches broad, and $2\frac{1}{2}$ thick; but including the half inch joint of mortar, when laid in brickwork, every dimension is to be counted half an inch more, making its length 9 inches, its breadth $4\frac{1}{2}$, and thickness 3 inches. So that every 4 courses of proper brickwork measures just 1 foot or 12 inches in height.

EXAMPLES.

Exam. 1. How many yards and rods of standard brickwork are in a wall whose length or compass is 57 feet 3 inches, and height 24 feet 6 inches; the wall being $2\frac{1}{2}$ bricks or 5 half bricks thick? Ans. 8 rods, $17\frac{2}{3}$ yards.

Exam. 2. Required the content of a wall 62 feet 6 inches long, and 14 feet 8 inches high, and $2\frac{1}{2}$ bricks thick?

Ans. 169·753 yards.

Exam. 3. A triangular gable is raised $17\frac{1}{2}$ feet high, on an end wall whose length is 24 feet 9 inches, the thickness being 2 bricks: required the reduced content?

Ans. $32{\cdot}08\frac{1}{3}$ yards.

Exam. 4. The end wall of a house is 28 feet 10 inches long, and 55 feet 8 inches high, to the eaves; 20 feet high is $2\frac{1}{2}$ bricks thick, other 20 feet high is 2 bricks thick, and the remaining 15 feet 8 inches is $1\frac{1}{2}$ brick thick; above which is a triangular gable, of 1 brick thick, which rises 42 courses of bricks, of which every 4 courses make a foot. What is the whole content in standard measure?

Ans. 253·626 yards.

IV. MASONS' WORK.

To Masonry belong all sorts of stone work; and the measure made use of is a foot, either superficial or solid.

Walls, columns, blocks of stone or marble, &c. are measured by the cubic foot; and pavements, slabs, chimney-pieces, &c. by the superficial or square foot.

Cubic or solid measure is used for the materials, and square measure for the workmanship.

In the solid measure, the true length, breadth, and thickness are taken and multiplied continually together. In the superficial, there must be taken the length and breadth of every part of the projection which is seen without the general upright face of the building.

EXAMPLES.

EXAM. 1. REQUIRED the solid content of a wall, 53 feet 6 inches long, 12 feet 3 inches high, and 2 feet thick?

Ans. $1310\frac{1}{4}$ feet.

EXAM. 2. What is the solid content of a wall, the length being 24 feet 3 inches, height 10 feet 9 inches, and 2 feet thick? Ans. 521·375 feet.

EXAM. 3. Required the value of a marble slab, at 8*s.* per foot; the length being 5 feet 7 inches, and breadth 1 foot 10 inches? Ans. 4*l.* 1*s.* $10\frac{1}{2}$*d.*

EXAM. 4. In a chimney-piece, suppose the

length of the mantle and slab, each	4 feet	6 inches
breadth of both together -	3	2
length of each jamb - -	4	4
breadth of both together -	1	9

Required the superficial content? Ans. 21 feet 10 inches.

V. CARPENTERS' AND JOINERS' WORK.

To this branch belongs all the wood-work of a house, such as flooring, partitioning, roofing, &c.

Large and plain articles are usually measured by the square foot or yard, &c.; but enriched mouldings, and some other articles, are often estimated by running or lineal measure; and some things are rated by the piece.

In measuring of Joists, take the dimensions of one joist,

and multiply it's content by the number of them; considering that each end is let into the wall about $\frac{2}{3}$ of the thickness, as it ought to be.

Partitions are measured from wall to wall for one dimension, and from floor to floor, as far as they extend, for the other.

The measure of Centering for Cellars is found by making a string pass over the surface of the arch for the breadth, and taking the length of the cellar for the length: but in groin centering, it is usual to allow double measure, on account of their extraordinary trouble.

In Roofing, the dimensions, as to length, breadth, and depth, are taken as in flooring joists, and the contents computed the same way.

In Floor-boarding, take the length of the room for one dimension, and the breadth for the other, to multiply together for the content.

For Stair-cases, take the breadth of all the steps, by making a line ply close over them, from the top to the bottom, and multiply the length of this line by the length of a step, for the whole area.—By the length of a step is meant the length of the front and the returns at the two ends; and by the breadth is to be understood the girts of its two outer surfaces, or the tread and riser.

For the Balustrade, take the whole length of the upper part of the hand-rail, and girt over its end till it meet the top of the newel-post, for the one dimension; and twice the length of the baluster on the landing, with the girt of the hand-rail, for the other dimension.

For Wainscoting, take the compass of the room for the one dimension; and the height from the floor to the ceiling, making the string ply close into all the mouldings, for the other.

For Doors, take the height and the breadth, to multiply them together for the area.—If the door be panneled on both sides, take double its measure for the workmanship; but if one side only be panneled, take the area and its half for the workmanship.—*For the Surrounding Architrave*, girt it about the uttermost part for its length; and measure over it, as far as it can be seen when the door is open, for the breadth.*

Window-shutters, Bases, &c. are measured in like manner.

In measuring of Joiners' work, the string is made to ply

close into all mouldings, and to every part of the work over which it passes.

EXAMPLES.

Exam. 1. Required the content of a floor, 48 feet 6 inches long, and 24 feet 3 inches broad? Ans. 11 sq. 76$\frac{1}{8}$ feet.

Exam. 2. A floor being 36 feet 3 inches long, and 16 feet 6 inches broad, how many squares are in it?

Ans. 5 sq. 98$\frac{1}{8}$ feet.

Exam. 3. How many squares are there in 173 feet 10 inches in length, and 10 feet 7 inches height, of partitioning?

Ans. 18·3973 squares.

Exam. 4. What cost the roofing of a house at 10*s*. 6*d*. a square; the length within the walls being 52 feet 8 inches, and the breadth 30 feet 6 inches; reckoning the roof $\frac{3}{2}$ of the flat? Ans. 12*l*. 12*s*. 11$\frac{3}{4}$*d*.

Exam. 5. To how much, at 6*s*. per square yard, amounts the wainscoting of a room; the height, taking in the cornice and mouldings, being 12 feet 6 inches, and the whole compass 83 feet 8 inches; also the three window-shutters are each 7 feet 8 inches by 3 feet 6 inches, and the door 7 feet by 3 feet 6 inches; the door and shutters, being worked on both sides, are reckoned work and half work?

Ans. 36*l*. 12*s*. 2$\frac{1}{2}$*d*.

VI. SLATERS' AND TILERS' WORK.

In these articles, the content of a roof is found by multiplying the length of the ridge by the girt over from eaves to eaves; making allowance in this girt for the double row of slates at the bottom, or for how much one row of slates or tiles is laid over another.

When the roof is of a true pitch, that is, forming a right angle at top; then the breadth of the building, with its half added, is the girt over both sides nearly.

In angles formed in a roof, running from the ridge to the eaves, when the angle bends inwards, it is called a valley; but when outwards, it is called a hip.

Deductions are made for chimney shafts or window holes.

EXAMPLES.

Exam. 1. Required the content of a slated roof, the length being 45 feet 9 inches, and the whole girt 34 feet 3 inches? Ans. $174\frac{5}{48}$ yards.

Exam. 2. To how much amounts the tiling of a house, at 25*s*. 6*d*. per square; the length being 43 feet 10 inches, and the breadth on the flat 27 feet 5 inches; also the eaves projecting 16 inches on each side, and the roof of a true pitch? Ans. 24*l*. 9*s*. $8\frac{1}{4}$*d*.

VII. PLASTERERS' WORK.

Plasterers' work is of two kinds; namely, ceiling, which is plastering on laths; and rendering, which is plastering on walls: which are measured separately.

The contents are estimated either by the foot or the yard, or the square, of 100 feet. Enriched mouldings, &c. are rated by running or lineal measure.

Deductions are made for chimneys, doors, windows, &c.

EXAMPLES.

Exam. 1. How many yards contains the ceiling which is 43 feet 3 inches long, and 25 feet 6 inches broad?

Ans. $122\frac{1}{2}$.

Exam. 2. To how much amounts the ceiling of a room, at 10*d*. per yard; the length being 21 feet 8 inches, and the breadth 14 feet 10 inches? Ans. 1*l*. 9*s*. $8\frac{3}{4}$*d*.

Exam. 3. The length of a room is 18 feet 6 inches, the breadth 12 feet 3 inches, and height 10 feet 6 inches; to how much amounts the ceiling and rendering, the former at 8*d*. and the latter at 3*d*. per yard; allowing for the door of 7 feet by 3 feet 8, and a fire-place of 5 feet square?

Ans. 1*l*. 13*s*. $3\frac{1}{2}$*d*.

Exam. 4. Required the quantity of plastering in a room, the length being 14 feet 5 inches, breadth 13 feet 2 inches, and height 9 feet 3 inches to the under side of the cornice, which girts $8\frac{1}{2}$ inches, and projects 5 inches from the wall on the upper part next the ceiling; deducting only for a door 7 feet by 4?

Ans. 53 yards	5 feet	$3\frac{1}{2}$ inches	of rendering
18	5	6	of ceiling
	39	$0\frac{1}{12}$	of cornice.

VIII. PAINTERS' WORK.

PAINTERS' work is computed in square yards. Every part is measured where the colour lies; and the measuring line is forced into all the mouldings and corners.

Windows are done at so much a piece. And it is usual to allow double measure for carved mouldings, &c.

EXAMPLES.

EXAM. 1. How many yards of painting contains the room which is 65 feet 6 inches in compass, and 12 feet 4 inches high? Ans. $89\frac{41}{54}$ yards.

EXAM. 2. The length of a room being 20 feet, its breadth 14 feet 6 inches, and height 10 feet 4 inches; how many yards of painting are in it, deducting a fire-place of 4 feet by 4 feet 4 inches, and two windows each 6 feet by 3 feet 2 inches? Ans. $73\frac{2}{27}$ yards.

EXAM. 3. What cost the painting of a room, at 6*d.* per yard; its length being 24 feet 6 inches, its breadth 16 feet 3 inches, and height 12 feet 9 inches; also the door is 7 feet by 3 feet 6, and the window-shutters to two windows each 7 feet 9 by 3 feet 6; but the breaks of the windows themselves are 8 feet 6 inches high, and 1 foot 3 inches deep; including also the window cills or seats, and the soffits above, the dimensions of which are known from the other dimensions: but deducting the fire-place of 5 feet by 5 feet 6? Ans. 3*l.* 3*s.* $10\frac{1}{4}$*d.*

IX. GLAZIERS' WORK.

GLAZIERS take their dimensions, either in feet, inches, and parts, or feet, tenths, and hundredths. And they compute their work in square feet.

In taking the length and breadth of a window, the cross bars between the squares are included. Also windows of round or oval forms are measured as square, measuring them to their greatest length and breadth, on account of the waste in cutting the glass.

EXAMPLES.

EXAM. 1. How many square feet contains the window which is 4·25 feet long, and 2·75 feet broad? Ans. $11\frac{2}{3}$.

EXAM. 2. What will the glazing a triangular sky-light come to, at 10*d*. per foot; the base being 12 feet 6 inches, and the perpendicular height 6 feet 9 inches?

Ans. 1*l*. 15*s*. $1\frac{3}{4}$*d*.

EXAM. 3. There is a house with three tiers of windows, three windows in each tier, their common breadth 3 feet 11 inches:

now the height of the first tier is	7 feet	10 inches
of the second	6	8
of the third	5	4

Required the expense of glazing at 14*d*. per foot?

Ans. 13*l*. 11*s*. $10\frac{1}{2}$*d*.

EXAM. 4. Required the expense of glazing the windows of a house at 13*d*. a foot; there being three stories, and three windows in each story:

the height of the lower tier is	7 feet	9 inches
of the middle	6	6
of the upper	5	$3\frac{1}{4}$
and of an oval window over the door	1	$10\frac{1}{2}$

the common breadth of all the windows being 3 feet 9 inches? Ans. 12*l*. 5*s*. 6*d*.

X. PAVERS' WORK.

PAVERS' work is done by the square yard. And the content is found by multiplying the length by the breadth.

EXAMPLES.

EXAM. 1. What cost the paving a foot-path, at 3*s*. 4*d*. a yard; the length being 35 feet 4 inches, and breadth 8 feet 3 inches? Ans. 5*l*. 7*s*. $11\frac{1}{2}$*d*.

EXAM. 2. What cost the paving a court, at 3*s*. 2*d*. per yard; the length being 27 feet 10 inches, and the breadth 14 feet 9 inches? Ans. 7*l*. 4*s*. $5\frac{1}{4}$*d*.

EXAM. 3. What will be the expense of paving a rectangular court-yard, whose length is 63 feet, and breadth 45 feet; in which there is laid a foot-path of 5 feet 3 inches broad, running the whole length, with broad stones, at 3*s*. a yard; the rest being paved with pebbles at 2*s*. 6*d*. a yard?

Ans. 40*l*. 5*s*. $10\frac{1}{2}$*d*.

XI. PLUMBERS' WORK.

Plumbers' work is rated at so much a pound, or else by the hundred weight of 112 pounds.

Sheet lead, used in roofing, guttering, &c. is from 6 to 10lb. to the square foot. And a pipe of an inch bore is commonly 13 or 14lb. to the yard in length.

EXAMPLES.

Exam. 1. How much weighs the lead which is 39 feet 6 inches long, and 3 feet 3 inches broad, at $8\frac{1}{2}$lb. to the square foot? Ans. $1091\frac{3}{16}$lb.

Exam. 2. What cost the covering and guttering a roof with lead, at 18*s*. the cwt; the length of the roof being 43 feet, and breadth or girt over it 32 feet; the guttering 57 feet long, and 2 feet wide; the former 9·831lb. and the latter 7·373lb. to the square foot? Ans. 115*l*. 9*s*. $1\frac{1}{2}$*d*.

XII. TIMBER MEASURING.

PROBLEM I.

To find the Area, or Superficial Content, of a Board or Plank.

Multiply the length by the mean breadth.

Note. When the board is tapering, add the breadths at the two ends together, and take half the sum for the mean breadth. Or else take the mean breadth in the middle.

By the Sliding Rule.

Set 12 on B to the breadth in inches on A; then against the length in feet on B, is the content on A, in feet and fractional parts.

EXAMPLES.

Exam. 1. What is the value of a plank, at $1\frac{1}{2}$*d*. per foot, whose length is 12 feet 6 inches, and mean breadth 11 inches? Ans. 1*s*. 5*d*.

EXAM. 2. Required the content of a board, whose length is 11 feet 2 inches, and breadth 1 foot 10 inches?

Ans. 20 feet 5 inches 8″.

EXAM. 3. What is the value of a plank, which is 12 feet 9 inches long, and 1 foot 3 inches broad, at $2\frac{1}{2}d.$ a foot?

Ans. $3s.\ 3\frac{1}{4}d.$

EXAM. 4. Required the value of 5 oaken planks at $3d.$ per foot, each of them being $17\frac{1}{2}$ feet long; and their several breadths as follows, namely, two of $13\frac{1}{2}$ inches in the middle, one of $14\frac{1}{2}$ inches in the middle, and the two remaining ones, each 18 inches at the broader end, and $11\frac{1}{4}$ at the narrower?

Ans. $1l.\ 5s.\ 9\frac{1}{2}d.$

PROBLEM II.

To find the Solid Content of Squared or Four-sided Timber.

MULTIPLY the mean breadth by the mean thickness, and the product again by the length, for the content nearly.

By the Sliding Rule.

C		D		D		C
As length	:	12 or 10	: :	quarter girt	:	solidity.

That is, as the length in feet on C, is to 12 on D, when the quarter girt is in inches, or to 10 on D, when it is in tenths of feet; so is the quarter girt on D, to the content on C.

Note 1. If the tree taper regularly from the one end to the other; either take the mean breadth and thickness in the middle, or take the dimensions at the two ends, and half their sum will be the mean dimensions: which multiplied as above, will give the content nearly.

2. If the piece do not taper regularly, but be unequally thick in some parts and small in others; take several different dimensions, add them all together, and divide their sum by the number of them, for the mean dimensions.

EXAMPLES.

EXAM. 1. The length of a piece of timber is 18 feet 6 inches, the breadths at the greater and less end 1 foot 6 inches and 1 foot 3 inches, and the thickness at the greater and less end 1 foot 3 inches and 1 foot; required the solid content?

Ans. 28 feet 7 inches.

EXAM. 2. What is the content of the piece of timber, whose length is $24\frac{1}{2}$ feet, and the mean breadth and thickness each 1·04 feet? Ans. $26\frac{1}{2}$ feet.

EXAM. 3. Required the content of a piece of timber, whose length is 20·38 feet, and its ends unequal squares, the side of the greater being $19\frac{1}{8}$ inches, and the side of the less $9\frac{7}{8}$ inches? Ans. 29·7562 feet.

EXAM. 4. Required the content of the piece of timber, whose length is 27·36 feet; at the greater end the breadth is 1·78, and thickness 1·23; and at the less end the breadth is 1·04, and thickness 0·91 feet? Ans. 41·278 feet.

PROBLEM III.

To find the Solidity of Round or Unsquared Timber.

MULTIPLY the square of the quarter girt, or of $\frac{1}{4}$ of the mean circumference, by the length, for the content.

By the Sliding Rule.

As the length upon C : 12 or 10 upon D : :
quarter girt, in 12ths or 10ths, on D : content on C.

Note 1. When the tree is tapering, take the mean dimensions as in the former problems, either by girting it in the middle, for the mean girt, or at the two ends, and taking half the sum of the two; or by girting it in several places, then adding all the girts together, and dividing the sum by the number of them, for the mean girt. But when the tree is very irregular, divide it into several lengths, and find the content of each part separately.

2. This rule, which is commonly used, gives the answer about $\frac{1}{4}$ less than the true quantity in the tree, or nearly what the quantity would be, after the tree is hewed square in the usual way: so that it seems intended to make an allowance for the squaring of the tree.

On this subject, however, Hutton's Mensuration, part v. sect. 4, may be advantageously consulted.

EXAMPLES.

EXAM. 1. A piece of round timber being 9 feet 6 inches long, and its mean quarter girt 42 inches; what is the content? Ans. $116\frac{3}{8}$ feet.

EXAM. 2. The length of a tree is 24 feet, its girt at the thicker end 14 feet, and at the smaller end 2 feet; required the content? Ans. 96 feet.

EXAM. 3. What is the content of a tree whose mean girt is 3·15 feet, and length 14 feet 6 inches?

Ans. 8·9922 feet.

EXAM. 4. Required the content of a tree; whose length is $17\frac{1}{4}$ feet, which girts in five different places as follows, namely, in the first place 9·43 feet, in the second 7·92, in the third 6·15, in the fourth 4·74, and in the fifth 3·16?

Ans. 42·519525.

PRACTICAL EXERCISES IN MENSURATION.

QUEST. 1. WHAT difference is there between a floor 28 feet long by 20 broad, and two others, each of half the dimensions; and what do all three come to at 45*s*. per square, or 100 square feet?

Ans. dif. 280 sq. feet. Amount 18 guineas.

QUEST. 2. An elm plank is 14 feet 3 inches long, and I would have just a square yard slit off it; at what distance from the edge must the line be struck? Ans. $7\frac{11}{19}$ inches.

QUEST. 3. A ceiling contains 114 yards 6 feet of plastering, and the room is 28 feet broad; what is the length of it?

Ans. $36\frac{6}{7}$ feet.

QUEST. 4. A common joist is 7 inches deep, and $2\frac{1}{2}$ thick; but I want a scantling just as big again, that shall be 3 inches thick; what will the other dimension be?

Ans. $11\frac{2}{3}$ inches.

QUEST. 5. A wooden trough cost me 3*s*. 2*d*. painting within, at 6*d* per yard; the length of it was 102 inches, and the depth 21 inches; what was the width?

Ans. $27\frac{3}{4}$ inches.

QUEST. 6. If my court-yard be 47 feet 9 inches square, and I have laid a foot-path with Purbeck-stone, of 4 feet wide, along one side of it; what will paving the rest with flints come to, at 6*d*. per square yard? Ans. 5*l*. 16*s*. $0\frac{1}{2}$*d*.

QUEST. 7. A ladder, 36 feet long, may be so planted, that it shall reach a window 30·7 feet from the ground on one side of the street; and, by only turning it over, without moving the foot out of its place, it will do the same by a window 18·9 feet high on the other side: what is the breadth of the street? Ans. 50·984 feet.

QUEST. 8. The paving of a triangular court, at 18*d*. per

foot, came to 100*l.*; the longest of the three sides was 88 feet; required the sum of the other two equal sides?

Ans. 10·685 feet.

QUEST. 9. The perambulator, or surveying wheel, is so contrived, as to turn just twice in the length of a pole, or $16\frac{1}{2}$ feet; required the diameter? Ans. 2·626 feet.

QUEST. 10. In turning a one-horse chaise within a ring of a certain diameter, it was observed, that the outer wheel made two turns, while the inner made but one: the wheels were both 4 feet high; and, supposing them fixed at the statutable distance of 5 feet asunder on the axle-tree, what was the circumference of the track described by the outer wheel? Ans. 62·832 feet.

QUEST. 11. What is the side of that equilateral triangle, whose area cost as much paving at 8*d.* a foot, as the pallisading the three sides did at a guinea a yard?

Ans. 72·746 feet.

QUEST. 12. A roof, which is 24 feet 8 inches by 14 feet 6 inches, is to be covered with lead at 8lb. per square foot: what will it come to at 18*s.* per cwt.? Ans. 22*l.* 19*s.* $10\frac{1}{4}$*d.*

QUEST. 13. Having a rectangular marble slab, 58 inches by 27, I would have a square foot cut off parallel to the shorter edge; I would then have the like quantity divided from the remainder parallel to the longer side; and this alternately repeated, till there shall not be the quantity of a foot left: what will be the dimensions of the remaining piece? Ans. 20·7 inches by 6·086.

N. B. This question may be solved neatly by an algebraical process, as may be seen in the Ladies' Diary for 1823.

QUEST. 14. Given two sides of an obtuse-angled triangle, which are 20 and 40 poles; required the third side, that the triangle may contain just an acre of land?

Ans. 58·876 or 23·099.

QUEST. 15. How many bricks will it take to build a wall, 10 feet high, and 500 feet long, of a brick and half thick; reckoning the brick 10 inches long, and 4 courses to the foot in height? Ans. 72000.

QUEST. 16. How many bricks will build a square pyramid of 100 feet on each side at the base, and also 100 feet perpendicular height: the dimensions of a brick being supposed 10 inches long, 5 inches broad, and 3 inches thick?

Ans. 3840000.

QUEST. 17. If, from a right-angled triangle, whose base

is 12, and perpendicular 16 feet, a line be drawn parallel to the perpendicular, cutting off a triangle whose area is 24 square feet; required the sides of this triangle?

Ans. 6, 8, and 10.

Quest. 18. If a round pillar, 7 inches across, have 4 feet of stone in it; of what diameter is the column, of equal length, that contains 10 times as much?

Ans. 22·136 inches.

Quest. 19. A circular fish-pond is to be made in a garden, that shall take up just half an acre; what must be the length of the chord that strikes the circle? Ans. $27\frac{3}{4}$ yards.

Quest. 20. When a roof is of a true pitch, the rafters are $\frac{3}{4}$ of the breadth of the building: now supposing the eaves-boards to project 10 inches on a side, what will the new ripping a house cost, that measures 32 feet 9 inches long, by 22 feet 9 inches broad on the flat, at 15*s.* per square? Ans. 8*l.* 15*s.* $9\frac{1}{2}$*d.*

Quest. 21. A cable, which is 3 feet long, and 9 inches in compass, weighs 22lb; what will a fathom of that cable weigh, which measures a foot round? Ans. $78\frac{2}{9}$lb.

Quest. 22. A plumber has put 28lb. per square foot into a cistern, 74 inches and twice the thickness of the lead long, 26 inches broad, and 40 deep; he has also put three stays across it within, 16 inches deep, of the same strength, and reckons 22*s.* per cwt. for work and materials. A mason has in return paved him a workshop, 22 feet 10 inches broad, with Purbeck stone, at 7*d.* per foot; and upon the balance finds there is 3*s.* 6*d.* due to the plumber; what was the length of the workshop, supposing sheet lead $\frac{1}{10}$ of an inch thick to weigh 5·899lbs. per foot? Ans. 32·2825 feet.

Quest. 23. The distance of the centres of two circles, whose diameters are each 50, being given, equal to 30; what is the area of the space inclosed by their circumferences?

Ans. 559·119.

Quest. 24. If 20 feet of iron railing weigh half a ton, when the bars are an inch and quarter square; what will 50 feet come to at $3\frac{1}{2}$*d.* per lb., the bars being but $\frac{7}{8}$ of an inch square? Ans. 20*l.* 0*s.* 2*d.*

Quest. 25. It is required to find the thickness of the lead in a pipe, of an inch and quarter bore, which weighs 14lb. per yard in length; the cubic foot of lead weighing 11325 ounces? Ans. ·20737 inches.

Quest. 26. Supposing the expense of paving a semicir-

cular plot, at 2*s*. 4*d*. per foot, come to 10*l*.; what is the diameter of it? Ans. 14·7737 feet.

QUEST. 27. What is the length of a chord which cuts off $\frac{1}{3}$ of the area from a circle whose diameter is 289?

Ans. 278·6716.

QUEST. 28. My plumber has set me up a cistern, and, his shop-book being burnt, he has no means of bringing in the charge, and I do not choose to take it down to have it weighed; but by measure he finds it contains $64\frac{3}{10}$ square feet, and that it is precisely $\frac{1}{8}$ of an inch in thickness. Lead was then wrought at 2*l*. per fother of $19\frac{1}{2}$ cwt. It is required from these items to make out the bill, allowing $6\frac{5}{9}$ oz. for the weight of a cubic inch of lead?

Ans. 4*l*. 11*s*. 2*d*.

QUEST. 29. What will the diameter of a globe be, when the solidity and superficial content are expressed by the same number? Ans. 6.

QUEST. 30. A sack, that would hold 3 bushels of corn, is $22\frac{1}{2}$ inches broad when empty; what will another sack contain, which, being of the same length, has twice its breadth or circumference? Ans. 12 bushels.

QUEST. 31. A carpenter is to put an oaken curb to a round well, at 8*d*. per foot square: the breadth of the curb is to be $7\frac{1}{4}$ inches, and the diameter within $3\frac{1}{2}$ feet: what will be the expense? Ans. 5*s*. $2\frac{1}{4}$*d*.

QUEST. 32. A gentleman has a garden 100 feet long, and 80 feet broad; and a gravel walk is to be made of an equal width half round it: what must the breadth of the walk be, to take up just half the ground? Ans. 25·968 feet.

QUEST. 33. The top of a may-pole, being broken off by a blast of wind, struck the ground at 15 feet distance from the foot of the pole; what was the height of the whole may-pole, supposing the length of the broken piece to be 39 feet? Ans. 75 feet.

QUEST. 34. Seven men bought a grinding-stone of 60 inches diameter, each paying $\frac{1}{7}$ part of the expense; what part of the diameter must each grind down for his share?

Ans. the 1st 4·4508, 2d 4·8400, 3d 5·3535, 4th 6·0765, 5th 7·2079, 6th 9·3935, 7th 22·6778 inches.

QUEST. 35. A maltster has a kiln, that is 16 feet 6 inches square: but he wants to pull it down, and build a new one, that may dry three times as much at once as the old one; what must be the length of its side? Ans. 28 feet, 7 inches.

QUEST. 36. How many 3 inch cubes may be cut out of a 12 inch cube? Ans. 64.

QUEST. 37. How long must the tether of a horse be, that will allow him to graze, quite around, just an acre of ground? Ans. $39\frac{1}{4}$ yards.

QUEST. 38. What will the painting of a conical spire come to at 8*d.* per yard; supposing the height to be 118 feet, and the circumference of the base 64 feet?

Ans. 14*l.* 0*s.* $8\frac{3}{4}$*d.*

QUEST. 39. The diameter of a standard corn bushel is $18\frac{1}{2}$ inches, and its depth 8 inches; then what must the diameter of that bushel be, whose depth is $7\frac{1}{2}$ inches?

Ans. 19·1067 inches.

QUEST. 40. Suppose the ball on the top of St. Paul's church is 6 feet in diameter; what did the gilding of it cost at $3\frac{1}{2}$ per square inch? Ans. 237*l.* 10*s.* 1*d.*

QUEST. 41. What will a frustum of a marble cone come to at 12*s.* per solid foot; the diameter of the greater end being 4 feet, that of the less end $1\frac{1}{2}$, and the length of the slant side 8 feet? Ans. 30*l.* 1*s.* $10\frac{1}{4}$*d.*

QUEST. 42. To divide a cone into three equal parts by sections parallel to the base, and to find the altitudes of the three parts, the height of the whole cone being 20 inches?

Ans. the upper part 13·867
the middle part 3·605
the lower part 2·528

QUEST. 43. A gentleman has a bowling-green, 300 feet long, and 200 feet broad, which he would raise 1 foot higher, by means of the earth to be dug out of a ditch that goes round it: to what depth must the ditch be dug, supposing its breadth to be every where 8 feet? Ans. $7\frac{21}{86}$ feet.

QUEST. 44. How high above the earth must a person be raised, that he may see $\frac{1}{3}$ of its surface?

Ans. to the height of the earth's diameter.

QUEST. 45. A cubic foot of brass is to be drawn into wire of $\frac{1}{40}$ of an inch in diameter; what will the length of the wire be, allowing no loss in the metal?

Ans. 97784·797 yards, or 55 miles 984·797 yards.

QUEST. 46. Of what diameter must the bore of a cannon be, which is cast for a ball of 24lb. weight, so that the diameter of the bore may be $\frac{1}{10}$ of an inch more than that of the ball? Ans. 5·647 inches.

Quest. 47. Supposing the diameter of an iron 9lb. ball to be 4 inches, as it is very nearly; it is required to find the diameters of the several balls weighing 1, 2, 3, 4, 6, 12, 18, 24, 32, 36, and 42lb., and the calibre of their guns, allowing $\frac{1}{50}$ of the calibre, or $\frac{1}{49}$ of the ball's diameter, for windage.

Answer.

Wt. ball.	Diameter ball.	Calibre gun.
1	1·9230	1·9622
2	2·4228	2·4723
3	2·7734	2·8301
4	3·0526	3·1149
6	3·4943	3·5656
9	4·0000	4·0816
12	4·4026	4·4924
18	5·0397	5·1425
24	5·[illegible]469	5·6601
32	6·1051	6·2297
36	6·3496	6·4792
42	6·6844	6·8208

Quest. 48. Supposing the windage of all mortars to be $\frac{1}{60}$ of the calibre, and the diameter of the hollow part of the shell to be $\frac{7}{10}$ of the calibre of the mortar: it is required to determine the diameter and weight of the shell, and the quantity or weight of powder requisite to fill it, for each of the several sorts of mortars, namely, the 13, 10, 8, 5·8, and 4·6 inch mortar.

Answer.

Calib. mort.	Diameter ball.	Wt. shell empty.	Wt. of powder.	Wt. shell filled.
4·6	4·523	8·320	0·583	8·903
5·8	5·703	16·677	1·168	17·845
8	7·867	43·764	3·065	46·829
10	9·833	85·476	5·986	91·462
13	12·783	187·791	13·151	200·942

QUEST. 49. If a heavy sphere, whose diameter is 4 inches, be let fall into a conical glass, full of water, whose diameter is 5, and altitude 6 inches; it is required to determine how much water will run over?

Ans. 26·272 cubic inches, or near $\frac{35}{47}$ parts of a pint.

QUEST. 50. The dimensions of a sphere and cone being the same as in the last question, and the cone only $\frac{1}{5}$ full of water; required what part of the axis of the sphere is immersed in the water? Ans. ·546 parts of an inch.

QUEST. 51. The cone being still the same, and $\frac{1}{5}$ full of water; required the diameter of a sphere which shall be just all covered by the water? Ans. 2·445996 inches.

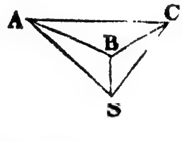
QUEST. 52. If a person, with an air balloon, ascend vertically from London, to such height that he can just see Oxford appear in the horizon; it is required to determine his height above the earth, supposing its circumference to be 25000 miles, and the distance between London and Oxford 49·5933 miles?

Ans. $\frac{311}{1000}$ of a mile, or 547 yards 1 foot.

QUEST. 53. In a garrison there are three remarkable objects, A, B, C, the distances of which from one to another are known to be, AB 213, AC 424, and BC 262 yards; I am desirous of knowing my position and distance at a place or station S, from whence I observed the angle ASB 13° 30′, and the angle CSB 29° 50′, both by geometry and trigonometry.

Answer.

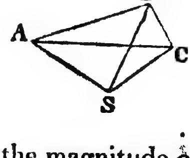
AS	605·7122,
BS	429·6814,
CS	521·2365.



QUEST. 54. Required the same as in the last question, when the point B is on the other side of AC, supposing AB 9, AC 12, and BC 6 furlongs; also the angle ASB 33° 45′, and the angle BSC 22° 30′.

Answer.

AS	10·64,
BS	15·64,
CS	14·01.



QUEST. 55. It is required to determine the magnitude of a cube of gold, of the standard fineness, which shall be

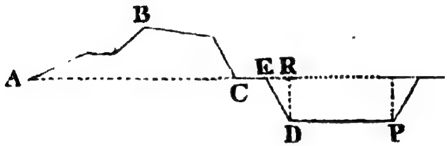
equal to a sum of 960 millions of pounds sterling; supposing a guinea to weigh 5 dwts $9\frac{1}{2}$ grains. Ans. 23·549 feet.

QUEST. 56. The ditch of a fortification is 1000 feet long, 9 feet deep, 20 feet broad at bottom, and 22 at top; how much water will fill the ditch?

Ans. 1158127 gallons nearly.

QUEST. 57. If the diameter of the earth be 7930 miles, and that of the moon 2160 miles: required the ratio of their surfaces, and also of their solidities: supposing them both to be globular, as they are very nearly?

Ans. the surfaces are as $13\frac{1}{2}$ to 1 nearly; and the solidities as $49\frac{1}{2}$ to 1 nearly.



QUEST. 58. Let ABC be the profile, or perpendicular section of a breast-work, and EP that of the ditch. Now, suppose the area of the section ABC is 88 feet, the depth of the ditch RD 6 feet, ER = SO = 3 feet; what is the breadth of the ditch at top when the sections of the ditch and the breast-work are equal; that is, when the earth thrown out of the ditch is sufficient to make the breast-work?

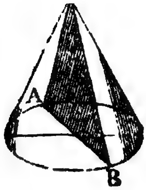
CONIC SECTIONS.

DEFINITIONS.

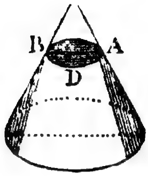
1. CONIC SECTIONS are the figures made by a plane cutting a cone.

2. According to the different positions of the cutting plane there arise five different figures or sections, namely, a triangle, a circle, an ellipsis, an hyperbola, and a parabola: the three last of which only are peculiarly called Conic Sections.

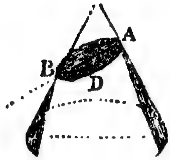
3. If the cutting plane pass through the vertex of the cone, and any part of the base, the section will evidently be a triangle; as VAB.



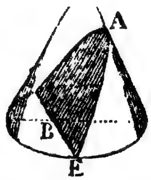
4. If the plane cut the cone parallel to the base, or make no angle with it, the section will be a circle; as ABD.



5. The section DAB is an ellipse when the cone is cut obliquely through both sides, or when the plane is inclined to the base in a less angle than the side of the cone is.

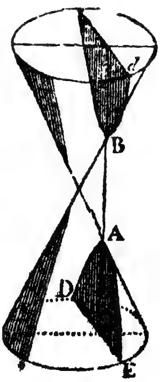


6. The section is a parabola, when the cone is cut by a plane parallel to the side, or when the cutting plane and the side of the cone make equal angles with the base.



7. The section is an hyperbola, when the cutting plane makes a greater angle with the base than the side of the cone makes.

8. And if all the sides of the cone be continued through the vertex, forming an opposite equal cone, and the plane be also continued to cut the opposite cone, this latter section will be the opposite hyperbola to the former; as *d*B*e*.



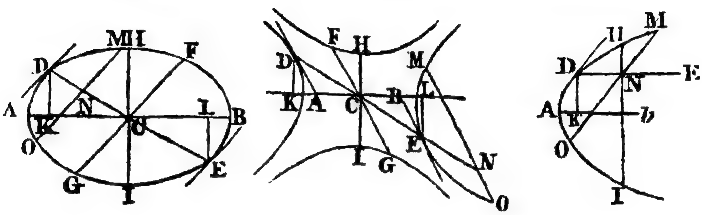
9. The Vertices of any section, are the points where the cutting plane meets the opposite sides of the cone, or the sides of the vertical triangular section; as A and B.

Hence the ellipse and the opposite hyperbolas, have each two vertices; but the parabola only one; unless we consider the other as at an infinite distance.

10. The Axis, or Transverse Diameter, of a conic section, is the line or distance AB between the vertices.

Hence the axis of a parabola is infinite in length, A*b* being only a part of it.

Ellipse. Hyperbolas. Parabola.



11. The centre C is the middle of the axis.

Hence the centre of a parabola is infinitely distant from the vertex. And of an ellipse, the axis and centre lie within the curve; but of an hyperbola, without.

12. A Diameter is any right line, as AB or DE, drawn through the centre, and terminated on each side by the curve; and the extremities of the diameter, or its intersections with the curve, are its vertices.

Hence all the diameters of a parabola are parallel to the axis, and infinite in length. Hence also every diameter of the ellipse and hyperbola has two vertices; but of the parabola, only one; unless we consider the other as at an infinite distance.

13. The Conjugate to any diameter, is the line drawn through the centre, and parallel to the tangent of the curve at the vertex of the diameter. So, FG, parallel to the tangent at D, is the conjugate to DE; and HI, parallel to the tangent at A, is the conjugate to AB.

Hence the conjugate HI, of the axis AB, is perpendicular to it.

14. An Ordinate to any diameter, is a line parallel to its conjugate, or to the tangent at its vertex, and terminated by the diameter and curve. So DK, EL, are ordinates to the axis AB; and MN, NO, ordinates to the diameter DE.

Hence the ordinates of the axis are perpendicular to it.

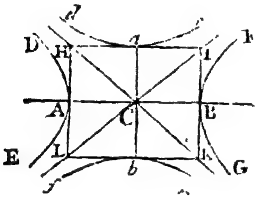
15. An Absciss is a part of any diameter contained between its vertex and an ordinate to it; as AK or BK, or DN or EN.

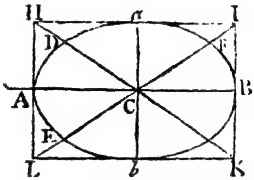
Hence, in the ellipse and hyperbola, every ordinate has two determinate abscisses; but in the parabola only one; the other vertex of the diameter being infinitely distant.

16. The Parameter of any diameter, is a third proportional to that diameter and its conjugate, in the ellipse and hyperbola, and to one absciss and its ordinate in the parabola.

17. The Focus is the point in the axis where the ordinate is equal to half the parameter. As K and L, where DK or EL is equal to the semi-parameter. The name focus being given to this point from the peculiar property of it mentioned in the corol. to theor. 9 in the Ellipse and Hyperbola following, and to theor. 6 in the Parabola.

Hence, the ellipse and hyperbola have each two foci; but the parabola only one.





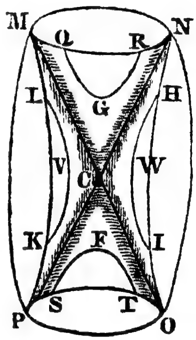
18. If DAE, FBG, be two opposite hyperbolas, having AB for their first or transverse axis, and *ab* for their second or conjugate axis. And if *dae*, *fbg*, be two other opposite hyperbolas having the same axes, but in the contrary order, namely, *ab* their first axis, and AB their second; then these two latter curves *dae*, *fbg*, are called the conjugate hyperbolas to the two former DAE, FBG; and each pair of opposite curves mutually conjugate to the other; being all for convenience of investigation referred to one plane, though they are only posited two and two in one plane; as will appear more evidently from the demonstration of th. 2. Hyperbola.

19. And if tangents be drawn to the four vertices of the curves, or extremities of the axes, forming the inscribed rectangle HIKL; the diagonals HCK, ICL, of this rectangle, are called the asymptotes of the curves. And if these asymptotes intersect at right angles, or the inscribed rectangle be a square, or the two axes AB and *ab* be equal, then the hyperbolas are said to be right-angled, or equilateral.

SCHOLIUM.

The rectangle inscribed between the four conjugate hyperbolas, is similar to a rectangle circumscribed about an ellipse, by drawing tangents in like manner, to the four extremities of the two axes; and the asymptotes or diagonals in the hyperbola, are analogous to those in the ellipse, cutting this curve in similar points, and making that pair of conjugate diameters which are equal to each other. Also, the whole figure formed by the four hyperbolas, is as it were, an ellipse turned inside out, cut open at the extremities, D, E, F, G, of the said equal conjugate diameters, and those four points drawn out to an infinite distance; the curvature being turned the contrary way, but the axes, and the rectangle passing through their extremities, continuing fixed.

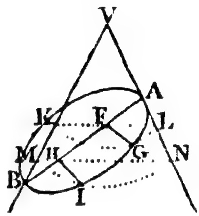
And further, if there be four cones CMN, COP, CMP, CNO, having all the same vertex C, and all their axes in the same plane, and their sides touching or coinciding in the common intersecting lines MCO, NCP; then if these four cones be all cut by one plane, parallel to the common plane of their axes, there will be formed the four hyperbolas, GQR, FST, VKL, WHI, of which each two opposites are equal; and each pair resembles the conjugates to the other two, as here in the annexed figure; but they are not accurately the conjugates, except only when the four cones are all equal, and then the four hyperbolic sections are all equal also.



OF THE ELLIPSE.

THEOREM I.

The Squares of the Ordinates of the Axis are to each other as the Rectangles of their Abscisses.



LET AVB be a plane passing through the axis of the cone; AGIH another section of the cone perpendicular to the plane of the former; AB the axis of this elliptic section; and FG, HI, ordinates perpendicular to it. Then it will be, as $FG^2 : HI^2 :: AF \cdot FB : AH \cdot HB$.

For, through the ordinates FG, HI, draw the circular sections KGL, MIN, parallel to the base of the cone, having KL, MN, for their diameters, to which FG, HI, are ordinates, as well as to the axis of the ellipse.

Now, by the similar triangles AFL, AHN, and BFK, BHM,

it is $AF : AH :: FL : HN$,

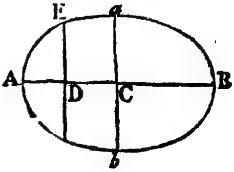
and $FB : HB :: KF : MH$;

hence, taking the rectangles of the corresponding terms, it is, the rect. $AF \cdot FB : AH \cdot HB :: KF \cdot FL : MH \cdot HN$.

But, by the circle, $KF \cdot FL = FG^2$, and $MH \cdot HN = HI^2$;
Therefore the rect. $AF \cdot FB : AH \cdot HB :: FG^2 : HI^2$. Q. E. D.

THEOREM II.

As the Square of the Transverse Axis
Is to the Square of the Conjugate :
So is the Rectangle of the Abscisses
To the Square of their Ordinate.



That is, $AB^2 : ab^2$ or $AC^2 : ac^2 :: AD \cdot DB : DE^2$.

For, by theor. 1, $AC \cdot CB : AD \cdot DB :: ca^2 : DE^2$;
But, if C be the centre, then $AC \cdot CB = AC^2$, and ca is the semi-conjugate.
Therefore $AC^2 : AD \cdot DB :: ac^2 : DE^2$;
or, by permutation, $AC^2 : ac^2 :: AD \cdot DB : DE^2$;
or, by doubling, $AB^2 : ab^2 :: AD \cdot DB : DE^2$. Q. E. D.

Corol. Or, by div. $AB : \frac{ab^2}{AB} :: AD \cdot DB$ or $CA^2 - CD^2 : DE^2$,

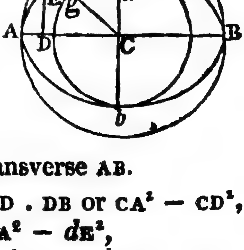
that is, $AB : p :: AD \cdot DB$ or $CA^2 - CD^2 : DE^2$;

where p is the parameter $\frac{ab^2}{AB}$, by the definition of it.

That is, As the transverse,
Is to its parameter,
So is the rectangle of the abscisses,
To the square of their ordinate.

THEOREM III.

As the Square of the Conjugate Axis :
Is to the Square of the Transverse Axis : :
So is the Rectangle of the Abscisses of the Conjugate, or the difference of the Squares of the Semi-conjugate and Distance of the Centre from any Ordinate of that Axis :
To the Square of their Ordinate.



That is,
$ca^2 : CB^2 :: ad \cdot db$ or $ca^2 - cd^2 : dE^2$.

For, draw the ordinate ED to the transverse AB.
Then, by theor. 1, $ca^2 : CA^2 :: DE^2 : AD \cdot DB$ or $CA^2 - CD^2$,
or - - - - - - $ca^2 : CA^2 :: cd^2 : CA^2 - dE^2$,
But - - - - - $ca^2 : CA^2 :: ca^2 : CA^2$,
theref. by subtr. $ca^2 : CA^2 :: ca^2 - cd^2$ or $ad \cdot db : dE^2$.
Q. E. D.

Corol. 1. If two circles be described on the two axes as diameters, the one inscribed within the ellipse, and the other circumscribed about it; then an ordinate in the circle will be to the corresponding ordinate in the ellipse, as the axis of this ordinate, is to the other axis.

That is, $CA : ca :: DG : DE$,
and $ca : CA :: dg : dE$.

For, by the nature of the circle, $AD . DB = DG^2$; theref. by the nature of the ellipse, $CA^2 : ca^2 :: AD . DB$ or $DG^2 : DE^2$,
or $CA : ca :: DG : DE$
In like manner - $ca : CA :: dg : dE$.
Also, by equality - $DG : DE$ or $cd :: dE$ or $DC : dg$.
Therefore cgG is a continued straight line.

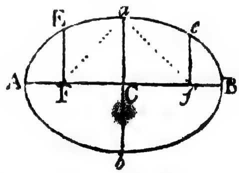
Corol. 2. Hence also, as the ellipse and circle are made up of the same number of corresponding ordinates, which are all in the same proportion of the two axes, it follows that the areas of the whole circle and ellipse, as also of any like parts of them, are in the same proportion of the two axes, or as the square of the diameter to the rectangle of the two axes; that is, the areas of the two circles, and of the ellipse, are as the square of each axis and the rectangle of the two: and therefore the ellipse is a mean proportional between the two circles.

THEOREM IV.

The Square of the Distance of the Focus from the Centre, is equal to the Difference of the Squares of the Semi-axes.

Or, the Square of the Distance between the Foci, is equal to the Difference of the Squares of the two Axes.

That is, $CF^2 = CA^2 - ca^2$
or $Ff^2 = AB^2 - ab^2$



For, to the focus F draw the ordinate FE; which, by the definition, will be the semi-parameter. Then, by the nature of the curve - - $CA^2 : ca^2 :: CA^2 - CF^2 : FE^2$;
and by the def. of the para. $CA^2 : ca^2 :: ca^2 : FE^2$;
therefore - - $ca^2 = CA^2 - CF^2$;
and by addit. and subtr. $CF^2 = CA^2 - ca^2$;
or, by doubling, - $Ff^2 = AB^2 - ab^2$. Q. E. D.

Corol. 1. The two semi-axes, and the focal distance from the centre, are the sides of a right-angled triangle CF*a*; and the distance F*a* from the focus to the extremity of the conjugate axis, is = AC the semi-transverse.

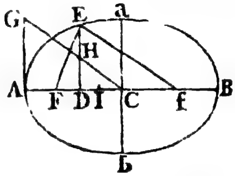
Corol. 2. The conjugate semi-axis C*a* is a mean proportional between AF, FB, or between A*f*, *f*B, the distances of either focus from the two vertices.

For $Ca^2 = CA^2 - CF^2 = (CA + CF) . (CA - CF) = AF . FB$.

THEOREM V.

The Sum of two Lines drawn from the two Foci to meet at any Point in the Curve, is equal to the Transverse Axis.

That is,
$FE + fE = AB$.



For, draw AG parallel and equal to C*a* the semi-conjugate; and join CG meeting the ordinate DE in H; also take CI a 4th proportional to CA, CF, CD.

Then by theor. 2, $CA^2 : AG^2 :: CA^2 - CD^2 : DE^2$;
and, by sim. tri. $CA^2 : AG^2 :: CA^2 - CD^2 : AG^2 - DH^2$;
consequently $DE^2 = AG^2 - DH^2 = Ca^2 - DH^2$.

Also, $FD = CF \quad CD$, and $FD^2 = CF^2 - 2CF . CD + CD^2$;
And, by right-angled triangles, $FE^2 = FD^2 + DE^2$;
therefore $FE^2 = CF^2 + Ca^2 - 2CF . CD + CD^2 - DH^2$;

But by theor. 4, $CF^2 + Ca^2 = CA^2$,
and by supposition, $2CF . CD = 2CA . CI$;
theref. $FE^2 = CA^2 - 2CA . CI + CD^2 - DH^2$.

Again, by supp. $CA^2 : CD^2 :: CF^2$ or $CA^2 - AG^2 : CI^2$;
and, by sim. tri. $CA^2 : CD^2 :: CA^2 - AG^2 : CD^2 - DH^2$;
therefore - $CI^2 = CD^2 - DH^2$;
consequently $FE^2 = CA^2 - 2CA . CI + CI^2$.

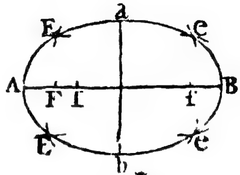
And the root or side of this square is $FE = CA - CI = AI$.

In the same manner it is found that $fE = CA + CI = BI$.
Conseq. by addit. $FE + fE = AI + BI = AB$. Q. E. D.

Corol. 1. Hence CI or CA − FE is a 4th proportional to CA, CF, CD.

Corol. 2. And fE − FE = 2CI; that is, the difference between two lines drawn from the foci, to any point in the curve, is double the 4th proportional to CA, CF, CD.

Corol. 3. Hence is derived the common method of describing this curve mechanically by points, or with a thread, thus:



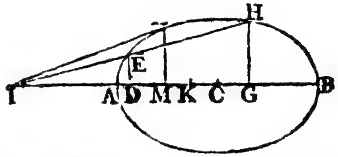
In the transverse take the foci F, f, and any point I. Then with the radii AI, BI, and centres F, f, describe arcs intersecting in E, which will be a point in the curve. In like manner, assuming other points I, as many other points will be found in the curve. Then with a steady hand, the curve line may be drawn through all the points of intersection E.

Or, take a thread of the length AB of the transverse axis, and fix its two ends in the foci F, f, by two pins. Then carry a pen or pencil round by the thread, keeping it always stretched, and its point will trace out the curve line.

THEOREM VI.

If from any Point I in the Axis produced, a Line IL be drawn touching the Curve in one Point L; and the Ordinate LM be drawn; and if C be the Centre or Middle of AB: Then shall CM be to CI as the Square of AM to the Square of AI.

That is,
$CM : CI :: AM^2 : AI^2$.



For, from the point I draw any other line IEH to cut the curve in two points E and H; from which let fall the perpendiculars ED and HG; and bisect DG in K.

Then, by theo. 1, $AD \cdot DB : AG \cdot GB :: DE^2 : GH^2$,
and by sim. triangles, $ID^2 : IG^2 :: DE^2 : GH^2$;
theref. by equality, $AD \cdot DB : AG \cdot GB :: ID^2 : IG^2$.

But $DB = CB + CD = AC + CD = AG + DC - CG = 2CK + AG$,
and $GB = CB - CG = AC - CG = AD + DC - CG = 2CK + AD$;
theref. $AD \cdot 2CK + AD \cdot AG : AG \cdot 2CK + AD \cdot AG :: ID^2 : IG^2$,
and, by div. $DG \cdot 2CK : IG^2 - ID^2$ or $DG \cdot 2IK :: AD \cdot 2CK + AD \cdot AG : ID^2$,

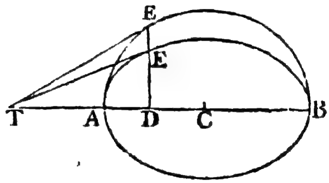
or $\quad 2CK : 2IK :: AD . 2CK + AD . AG : ID^2$,
or $AD . 2CK : AD . 2IK :: AD . 2CK + AD . AG : ID^2$;
theref. by div. $CK : IK :: AD . AG : ID^2 - AD . 2IK$,
and, by comp. $CK : IC :: AD \quad AG : ID^2 - AD . ID + IA$,
or $\quad CK : CI \quad AD \quad AG : AI^2$.

But, when the line IH, by revolving about the point I, comes into the position of the tangent IL, then the points E and H meet in the point L, and the points D, K, G, coincide with the point M; and then the last proportion becomes $CM : CI :: AM^2 : AI^2$. Q. E. D.

THEOREM VII.

If a Tangent and Ordinate be drawn from any Point in the Curve, meeting the Transverse Axis; the Semi-transverse will be a Mean Proportional between the Distances of the said Two Intersections from the Centre.

That is, CA is a mean proportional between CD and CT; or CD, CA, CT, are continued proportionals.



For, by theor. 6, $CD : CT :: AD^2 : AT^2$
that is, $CD \quad CT :: (CA - CD)^2 : (CT - CA)^2$,
or - $CD \quad CT :: CD^2 + CA^2 : CA^2 + CT^2$,
and - $CD \quad DT :: CD^2 + CA^2 : CT^2 - CD^2$,
or - $CD \quad DT :: CD^2 + CA^2 : (CT + CD)DT$,
or - $CD^2 \quad CD . DT :: CD^2 + CA^2 : (CD . DT) + (CT . DT)$,
hence $CD^2 \quad CA^2 : CD . DT : CT . DT$,
and - $CD^2 \quad CA^2 :: CD : CT$.
therefore (th. 78, Geom.) $CD : CA :: CA : CT$. Q. E. D.

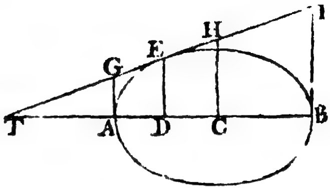
Corol. 1. Since CT is always a third proportional to CD, CA; if the points D, A, remain constant, then will the point T be constant also; and therefore all the tangents will meet in this point T, which are drawn from the point E, of every ellipse described on the same axis AB, where they are cut by the common ordinate DEE drawn from the point D.

Corol. 2. When the outer ellipse, by enlarging, becomes a circle, as at the upper figure at E, then by drawing ET perp. to CE, and joining T to the lower E, the tangent to the point E at the ellipse is obtained.

THEOREM VIII.

If there be any Tangent meeting Four Perpendiculars to the Axis drawn from these four Points, namely, the Centre, the two Extremities of the Axis, and the Point of Contact; those Four Perpendiculars will be Proportionals.

That is,
AG : DE : : CH : BI.



For, by theor. 7, TC : AC : : AC : DC,
theref. by div. TA : AD : : TC : AC or CB,
and by comp. TA : TD : : TC : TB,
and by sim. tri. ∴ AG : DE : : CH : BI. Q. E. D.

Corol. 1. Hence TA, TD, TC, TB and TG, TE, TH, TI } are also proportionals.
For these are as AG, DE, CH, BI, by similar triangles.

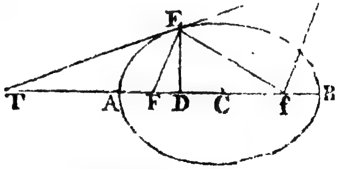
Corol. 2. Draw AI to cut DE in P; then since TA : TE : : TC : TI, the triangles TAE, TCI are similar, as well as the triangles AED, CBI, and ADP, ABI.

Hence - AD : DE : : CB : BI
and - AD : DP : : AB : BI
∴ DE : DP : : AB : CB : : 2 : 1; which suggests another simple practical method of drawing a tangent to an ellipse.

THEOREM IX.

If there be any Tangent, and two Lines drawn from the Foci to the Point of Contact; these two lines will make equal Angles with the Tangent.

That is,
the $\angle$FET = $\angle f$Ec.



For, draw the ordinate DE and fe parallel to FE.
By cor. 1, theor. 5, CA : CD : : CF : CA — FE,
and by theor. 7, CA : CD : : CT : CA;
therefore CT : CF : : CA : CA — FE;
and by add. and sub. TF : Tf : : FE : 2CA − FE or fE by th. 5.
But by sim. tri. TF : Tf : : FE : fe;

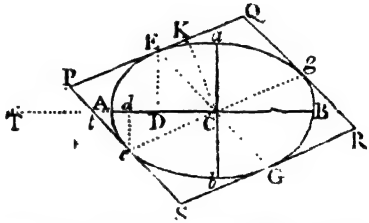
therefore *f*E = *fe*, and conseq. ∠ *e* = ∠ *f*E*e*.
But because FE is parallel to *fe*, the ∠ *e* = ∠ FET;
therefore ∠ FET = ∠ *f*E*e*. Q. E. D.

Corol. As opticians find that the angle of incidence is equal to the angle of reflection, it appears from this theorem, that rays of light issuing from the one focus, and meeting the curve in every point, will be reflected into lines drawn from those points to the other focus. So the ray *f*E is reflected into FE. And this is the reason why the points F, *f*, are called the *foci*, or burning points.

THEOREM X.

All the Parallelograms circumscribed about an Ellipse are equal to one another, and each equal to the Rectangle of the two Axes.

That is,
the parallelogram PQRS =
the rectangle AB . *ab*.



Let EG, *eg*, be two conjugate diameters parallel to the sides of the parallelogram, and dividing it into four less and equal parallelograms. Also, draw the ordinates DE, *de*, and CK perpendicular to PQ; and let the axis CA produced meet the sides of the parallelogram, produced if necessary, in T and *t*.

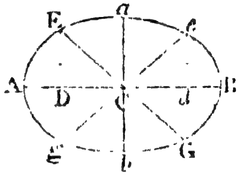
Then, by theor. 7, CT : CA :: CA : CD,
and - - C*t* : CA :: CA : C*d*;
theref. by equality, CT : C*t* :: C*d* : CD;
but, by sim. triangles, CT : C*t* :: TD : C*d*,
theref. by equality, TD : C*d* :: C*d* : CD,
and the rectangle TD . DC is = the square Cd^2.
Again, by theor. 7, CD CA :: CA : CT,
or, by division, CD CA :: DA : AT,
and by composition, CD DB :: AD : DT;
conseq. the rectangle CD DT = Cd^2 = AD . DB *.
But, by theor. 1, CA^2 Ca^2 :: (AD . DB or) Cd^2 : DE^2,
therefore CA C*a* :: C*d* : DE;
or C*a* DE :: CA : C*d*;

* *Corol.* Because Cd^2 = AD . DB = $CA^2 - CD^2$,
therefore $CA^2 = CD^2 + Cd^2$.
In like manner, $Ca^2 = DE^2 + de^2$.

By th. 7 $ct : CA :: CA : cd$,
by equality - $ct : CA :: ca : DE$,
by sim. tri. - $ct : CT :: dc : DE$,
theref. by equality, $CT : CA :: ca : de$.
But, by sim. tri. $CT : CK :: ce : dc$;
theref. by equality, $CK : CA :: ca : ce$,
and the rectangle $CK \cdot ce = CA \cdot ca$.
But the rect. $CK \cdot ce$ = the parallelogram CEPe,
theref. the rect. $CA \cdot ca$ = the parallelogram CEPe,
conseq. the rect. $AB \cdot ab$ = the parallelogram PQRS. Q. E. D.

THEOREM XI.

The Sum of the Squares of every Pair of Conjugate Diameters, is equal to the same constant Quantity, namely, the Sum of the Squares of the two Axes.



That is,
$AB^2 + ab^2 = EG^2 + eg^2$;
where EG, eg, are any pair of conjugate diameters.

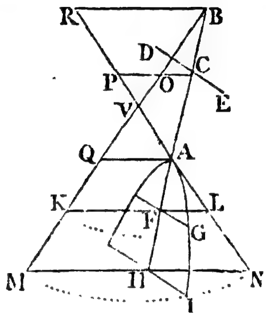
For, draw the ordinates ED, cd.
Then, by cor. to Theor. 10, $CA^2 = CD^2 + cd^2$,
and - - - $ca^2 = DE^2 + de^2$;
therefore the sum $CA^2 + ca^2 = CD^2 + DE^2 + cd^2 + de^2$.
But, by right-angled Δs, $CE^2 = CD^2 + DE^2$,
and - - - $ce^2 = cd^2 + de^2$;
therefore the sum $CE^2 + ce^2 = CD^2 + DE^2 + cd^2 + de^2$.
consequently - $CA^2 + ca^2 = CE^2 + ce^2$;
or, by doubling, $AB^2 + ab^2 = EG^2 + eg^2$. Q. E. D.

Note. All these theorems in the Ellipse, and their demonstrations, are the very same, word for word, as the corresponding number of those in the Hyperbola, next following, having only sometimes the word *sum* changed for the word *difference*.

OF THE HYPERBOLA.

THEOREM I.

The Squares of the Ordinates of the Axis are to each other as the Rectangles of their Abscisses.



Let AVB be a plane passing through the vertex and axis of the opposite cones; AGIH another section of them perpendicular to the plane of the former; AB the axis of the hyperbolic sections; and FG, HI, ordinates perpendicular to it. Then it will be, as $FG^2 : HI^2 :: AF . FB : AH . HB$.

For, through the ordinates FG, HI, draw the circular sections KGL, MIN, parallel to the base of the cone, having KL, MN, for their diameters, to which FG, HI, are ordinates, as well as to the axis of the hyperbola.

Now, by the similar triangles AFL, AHN, and BFK, BHM,

it is $AF : AH :: FL : HN$,

and $FB : HB :: KF : MH$;

hence, taking the rectangles of the corresponding terms,

it is, the rect. $AF . FB : AH . HB :: KF . FL : MH . HN$.

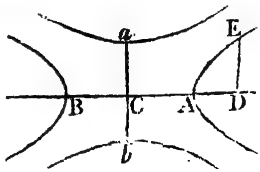
But, by the circle, $KF . FL = FG^2$, and $MH . HN = HI^2$;

Therefore the rect. $AF . FB : AH . HB :: FG^2 : HI^2$.

Q. E. D.

THEOREM II.

As the Square of the Transverse Axis
Is to the Square of the Conjugate :
So is the Rectangle of the Abscisses
To the Square of their Ordinate.



That is, $AB^2 : ab^2$ or
$AC^2 : ac^2$ AD . DB

For, by theor. 1, $CA \cdot CB : AD \cdot DB :: Ca^2 : DE^2$;
But, if C be the centre, then $AC \cdot CB = AC^2$, and Ca is the semi-conj.

Therefore - $AC^2 : AD \cdot DB :: ac^2 : DE^2$;
or, by permutation, $AC^2 : ac^2 :: AD \cdot DB : DE^2$;
or, by doubling, $AB^2 : ab^2 :: AD \cdot DB : DE^2$. Q. E. D.

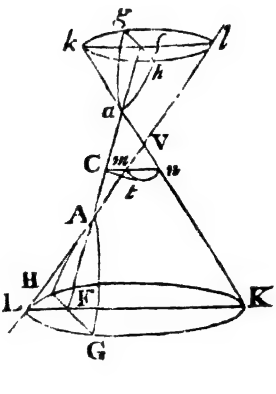
Corol. Or, by div. $AB : \frac{ab^2}{AB} :: AD \cdot DB$ or $CD^2 - CA^2 : DE^2$,

that is, $AB : p :: AD \cdot DB$ or $CD^2 - CA^2 : DE^2$;

where p is the parameter $\frac{ab^2}{AB}$ by the definition of it.

That is, As the transverse,
Is to its parameter,
So is the rectangle of the abscisses,
To the square of their ordinate.

Otherwise, thus:



Let a continued plane, cut from the two opposite cones, the two mutually connected opposite hyperbolas HAG, *hag*, whose vertices are A, *a*, and bases HG, *hg*, parallel to each other, falling in the planes of the two parallel circles LGK, *lgk*. Through C, the middle point of A*a*, let a plane be drawn parallel to that of LGK, it will cut in the cone LVK a circular section, whose diameter is *mn*; to which circular section, let C*t* be a tangent at *t*.

Then, by sim. tri. ACm, AFL $\quad AC : Cm \quad AF : FL$;

and, by sim. tri. aCn, aFK $\quad aC : Cn \quad aF : FK$.

$\therefore AC \cdot Ca : Cm \cdot Cn : AF \cdot Fa : LF \quad FK$,

or, $AC^2 : Ct^2 \quad AF \cdot Fa : FG^2$.

In like manner, for the opposite hyperbola

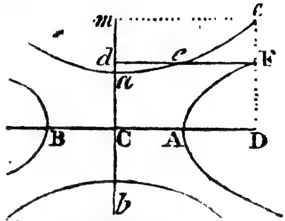
$AC^2 : Ct^2 :: Af \cdot fa : fg^2$.

Here C*t* is what is usually denominated the semi-conjugate to the opposite hyperbolas HAK, *hak*: but it is evidently *not in the same plane* with them.

THEOREM III.

As the Square of the Conjugate Axis :
To the Square of the Transverse Axis : :
The Sum of the Squares of the Semi-conjugate, and
Distance of the Centre from any Ordinate of the Axis :
The Square of their Ordinate.

That is,
$ca^2 : CA^2 :: ca^2 + cd^2 : dE^2$.



For, draw the ordinate ED to the transverse AB.

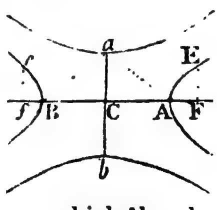
Then, by theor. 1, $ca^2 : CA^2 :: DE^2 : AD \cdot DB$ or $CD^2 - CA^2$,
or - - $ca^2 : CA^2 :: cd^2 : dE^2 - CA^2$.
But - - $ca^2 : CA^2 :: ca^2 : CA^2$.
theref. by compos. $ca^2 : CA^2 :: CA^2 + cd^2 : dE^2$.
In like manner, $CA^2 : ca^2 :: CA^2 + CD^2 : De^2$. Q. E. D.

Corol. By the last theor. $CA^2 : ca^2 :: CD^2 - CA^2 : DE^2$,
and by this theor. $CA^2 : ca^2 :: CD^2 + CA^2 : De^2$,
therefore - $DE^2 : De^2 :: CD^2 - CA^2 : CD^2 + CA^2$.
In like manner, $De^2 : dE^2 :: cd^2 - ca^2 : CD^2 + ca^2$.

THEOREM IV.

The Square of the Distance of the Focus from the Centre, is equal to the Sum of the Squares of the Semi-axes.
Or, the Square of the Distance between the Foci, is equal to the Sum of the Squares of the two Axes.

That is,
$CF^2 = CA^2 + ca^2$, or
$Ff^2 = AB^2 + ab^2$.



For, to the focus F draw the ordinate FE; which, by the definition, will be the semi-parameter. Then, by the nature
of the curve - $CA^2 : ca^2 :: CF^2 - CA^2 : FE^2$;
and by the def. of the para. $CA^2 : ca^2 :: ca^2 : FE^2$;
therefore - - $ca^2 = CF^2 - CA^2$;
and by addition, - $CF^2 = CA^2 + ca^2$;
or, by doubling, - $Ff^2 = AB^2 + ab^2$. Q. E. D.

Corol. 1. The two semi-axes, and the focal distance from the centre, are the sides of a right-angled triangle CAa; and the distance Aa is $=$ CF the focal distance.

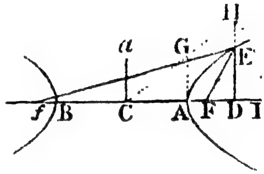
Corol. 2. The conjugate semi-axis Ca is a mean proportional between AF, FB, or between Af, fB, the distances of either focus from the two vertices.

For $Ca^2 = CF^2 - CA^2 = (CF + CA) \cdot (CF - CA) = AF \cdot FB$.

THEOREM V.

The Difference of two Lines drawn from the two Foci, to meet at any Point in the Curve, is equal to the Transverse Axis.

That is,
$f\text{E} - \text{FE} = \text{AB}$.



For, draw AG parallel and equal to Ca the semi-conjugate; and join CG, meeting the ordinate DE produced in H; also take CI a 4th proportional to CA, CF, CD.

Then, by th. 2, $CA^2 : AG^2 :: CD^2 - CA^2 : DE^2$;
and, by sim. Δs, $CA^2 : AG^2 :: CD^2 - CA^2 : DH^2 - AG^2$;
consequently $DE^2 = DH^2 - AG^2 = DH^2 - Ca^2$.
Also, $FD = CF \sim CD$, and $FD^2 = CF^2 - 2CF \cdot CD + CD^2$;
and, by right-angled triangles, $FE^2 = FD^2 + DE^2$.
therefore $FE^2 = CF^2 - Ca^2 - 2CF \cdot CD + CD^2 + DH^2$.
But, by theor. 4, $CF^2 - Ca^2 = CA^2$,
and, by supposition, $2CF \cdot CD = 2CA \cdot CI$;
theref. $FE^2 = CA^2 - 2CA \cdot CI + CD^2 + DH^2$;
Again, by suppos. $CA^2 : CD^2 :: CF^2$ or $CA^2 + AG^2 : CI^2$;
and, by sim. tri. $CA^2 : CD^2 :: CA^2 + AG^2 : CD^2 + DH^2$;
therefore - $CI^2 = CD^2 + DH^2 = CH^2$;
consequently $FE^2 = CA^2 - 2CA \cdot CI + CI^2$.
And the root or side of this square is $FE = CI - CA = AI$.
In the same manner, it is found that $f\text{E} = \text{CI} + \text{CA} = \text{BI}$.
Conseq. by subtract. $f\text{E} - \text{FE} = \text{BI} - \text{AI} = \text{AB}$. Q. E. D.

Corol. 1. Hence $CH = CI$ is a 4th proportional to CA, CF, CD.

Corol. 2. And $f\text{E} + \text{FE} = 2\text{CH}$ or 2CI; or FE, CH, fE, are in continued arithmetical progression, the common difference being CA the semi-transverse.

Corol. 3. Hence is derived the common method of describing this curve mechanically by points, thus:

In the transverse AB, produced, take the foci F, *f*, and any point I. Then with the radii AI, BI, and centres F, *f*, describe arcs intersecting in E, which will be a point in the curve. In like manner, assuming other points I, as many other points will be found in the curve.

Then, with a steady hand, the curve line may be drawn through all the points of intersection E.

In the same manner are constructed the other two or conjugate hyperbolas, using the axis *ab* instead of AB.

THEOREM VI.

If from any Point I in the Axis, a Line IL be drawn touching the Curve in one Point L; and the Ordinate LM be drawn; and if C be the Centre or the Middle of AB: Then shall CM be to CI as the Square of AM to the Square of AI.

That is,

$CM : CI :: AM^2 : AI^2$.

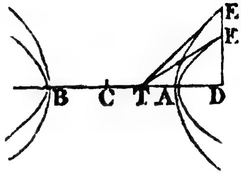
For, from the point I draw any line IEH to cut the curve in two points E and H; from which let fall the perps. ED, HG; and bisect DG in K.

Then, by theor. 1, $AD \cdot DB : AG \cdot GB :: DE^2 : GH^2$,
and by sim. triangles, $ID^2 : IG^2 :: DE^2 : GH^2$;
theref. by equality, $AD \cdot DB : AG \cdot GB :: ID^2 : IG^2$.
But $DB = CB + CD = CA + CD = CG + CD - AG = 2CK - AG$,
and $GB = CB + CG = CA + CG = CG + CD - AD = 2CK - AD$;
theref. $AD \cdot 2CK - AD \cdot AG : AG \cdot 2CK - AD \cdot AG :: ID^2 : IG^2$,
and, by div. $DG \cdot 2CK : IG^2 - ID^2$ or $DG \cdot 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$.
or - $2CK : 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$;
or $AD \cdot 2CK : AD \cdot 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$;
theref. by div. $CK : IK : AD \cdot AG : AD \cdot 2IK - ID^2$,
and, by div. $CK : CI \quad AD \cdot AG : ID^2 - AD \cdot (ID + IA)$,
or - $CK : CI :: AD \cdot AG : AI^2$.

But, when the line IH, by revolving about the point I, comes into the position of the tangent IL, then the points E and H meet in the point L, and the points D, K, G, coincide with the point M; and then the last proportion becomes $CM : CI :: AM^2 : AI^2$. Q. E. D.

THEOREM VII.

If a Tangent and Ordinate be drawn from any Point in the Curve, meeting the Transverse Axis; the Semi-transverse will be a Mean Proportional between the Distances of the said Two Intersections from the Centre.



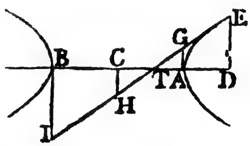
That is,
CA is a mean proportional between CD and CT; or CD, CA, CT, are continued proportionals.

For, by th. 6, $CD : CT :: AD^2 : AT^2$,
that is, $CD : CT :: (CD - CA)^2 : (CA - CT)^2$,
or $CD : CT :: CD^2 + CA^2 : CA^2 + CT^2$,
and $CD : DT :: CD^2 + CA^2 : CD^2 - CT^2$,
or $CD : DT :: CD^2 + CA^2 : (CD + CT)\ DT$,
or $CD^2 : CD . DT :: CD^2 + CA^2 : CD . DT + CT . TD$;
hence $CD^2 : CA :: CD . DT : CT . TD$,
and $CD^2 : CA^2 :: CD : CT$,
theref. (th. 78, Geom.) $CD : CA :: CA : CT$. Q. E. D.

Corol. Since CT is always a third proportional to CD, CA; if the points D, A, remain constant, then will the point T be constant also; and therefore all the tangents will meet in this point T, which are drawn from the point E, of every hyperbola described on the same axis AB, where they are cut by the common ordinate DEE drawn from the point D.

THEOREM VIII.

If there be any Tangent meeting Four Perpendiculars to the Axis drawn from these four Points, namely, the Centre, the two Extremities of the Axis, and the Point of Contact; those Four Perpendiculars will be Proportionals.



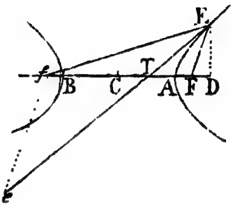
That is,
$AG : DE :: CH : BI$.

For, by theor. 7, $TC : AC :: AC : DC$,
theref. by div. $TA : AD :: TC : AC$ or CB,
and by comp. $TA : TD :: TC : TB$,
and by sim. tri. $AG : DE :: CH : BI$. Q. E. D.

Corol. Hence TA, TD, TC, TB and TG, TE, TH, TI are also proportionals.
For these are as AG, DE, CH, BI, by similar triangles.

THEOREM IX.

If there be any Tangent, and two Lines drawn from the Foci to the Point of Contact; these two Lines will make equal Angles with the Tangent.



That is,
the $\angle FET = \angle fEe$.

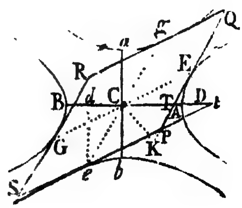
For, draw the ordinate DE, and fe parallel to FE.

By cor. 1, theor. 5, $CA : CD :: CF : CA + FE$,
and by th. 7, $CA : CD :: CT : CA$;
therefore $CT : CF :: CA : CA + FE$;
and by add. and sub. $TF : Tf :: FE : 2CA + FE$ or fE by th. 5.
But by sim. tri. $TF : Tf :: FE : fE$;
therefore $fE = fe$, and conseq. $\angle e = \angle fEe$.
But, because FE is parallel to fe, the $\angle e = \angle FET$;
therefore the $\angle FET = \angle fEe$. Q. E. D.

Corol. As opticians find that the angle of incidence is equal to the angle of reflection, it appears, from this proposition, that rays of light issuing from the one focus, and meeting the curve in every point, will be reflected into lines drawn from the other focus. So the ray fE is reflected into FE. And this is the reason why the points F, f, are called *foci*, or burning points.

THEOREM X.

All the Parallelograms inscribed between the four Conjugate Hyperbolas are equal to one another, and each equal to the Rectangle of the two Axes.



That is,
the parallelogram $PQRS$ = the rectangle $AB \cdot ab$.

Let EG, cg, be two conjugate diameters parallel to the sides of the parallelogram, and dividing it into four less and equal parallelograms. Also, draw the ordinates DE, de, and CK perpendicular to PQ; and let the axis produced meet the sides of the parallelograms, produced, if necessary, in T and t.

Then, by theor. 7,	$CT : CA :: CA : CD$,
and - -	$ct : CA :: CA : cd$;
theref. by equality,	$CT : ct :: cd : CD$;
but, by sim. triangles,	$CT : ct :: TD : cd$,
theref. by equality,	$TD : cd :: cd : CD$,
and the rectangle	$TD \cdot DC$ is $=$ the square cd^2.
Again, by theor. 7,	$CD : CA :: CA : CT$,
or, by division,	$CD : CA :: DA : AT$,
and, by composition,	$CD : DB :: DA : DT$,
conseq. the rectangle	$CD \cdot DT :: cd^2 = AD \cdot DB$ *
But, by theor. 1,	$CA^2 : ca^2 ::$ ($AD \cdot DB$ or) $cd^2 : DE^2$,
therefore -	$CA : ca :: cd : DE$,
or - -	$ca : DE :: CA : cd$.
By theor. 7, -	$CA : ct :: cd : CA$.
By equality -	$ct : CA :: ca : DE$.
By sim. tri. -	$ct : CT :: de : DE$;
theref. by equality,	$CT : CA :: ca : de$.
But, by sim. tri.	$CT : CK :: ce : de$;
theref. by equality,	$CK : CA :: ca : ce$.
and the rectangle	$CK \cdot ce = CA \cdot ca$.
But the rect.	$CK \cdot ce =$ the parallelogram CEPe,
theref. the rect.	$CA \cdot ce =$ the parallelogram CEPe,
conseq. the rect.	$AB \cdot ab =$ the paral. PQRS. Q. E. D.

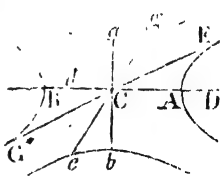
THEOREM XI.

The Difference of the Squares of every Pair of Conjugate Diameters, is equal to the same constant Quantity, namely the Difference of the Squares of the two Axes.

That is,

$$AB^2 - ab^2 = EG^2 - cg^2;$$

where EG, cg are any conjugate diameters.



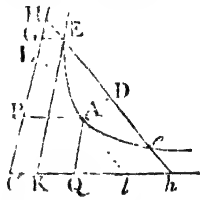
* *Corol.* Because $cd^2 = AD \cdot DB = CD^2 - CA^2$,
therefore $CA^2 = CD^2 - cd^2$.
In like manner $ca^2 = de^2 - DE^2$

For, draw the ordinates ED, ed.

Then, by cor. to theor. 10, $CA^2 = CD^2 - cd^2$,
and - - - - - $ca^2 = de^2 - DE^2$;
theref. the difference $CA^2 - ca^2 = CD^2 + DE^2 - cd^2 - de^2$.
But, by right-angled $\triangle$s, $CE^2 = CD^2 + DE^2$,
and - - - - - $ce^2 = cd^2 + de^2$;
theref. the difference $CE^2 - ce^2 = CD^2 + DE^2 - cd^2 - de^2$.
consequently - $CA^2 - ca^2 = CE^2 - CE^2$;
or, by doubling, $AB^2 - ab^2 = EG^2 - eg^2$. Q. E. D.

THEOREM XII.

All the Parallelograms are equal which are formed between the Asymptotes and Curve, by Lines drawn Parallel to the Asymptotes.



That is, the lines GE, EK, AP, AQ, being parallel to the asymptotes CH, Cl; then the paral. CGEK = paral. CPAQ.

For, let A be the vertex of the curve, or extremity of the semi-transverse axis AC, perp. to which draw AL or Al, which will be equal to the semi-conjugate, by definition 19. Also, draw HEDeh parallel to Ll,

Then, by theor. 2, $CA^2 : AL^2 :: CD^2 - CA^2 : DE^2$,
and, by parallels, $CA^2 : AL^2 :: CD^2 : DH^2$;
theref. by subtract. $CA^2 : AL^2 :: CA^2 : DH^2 - DE^2$ or rect. $HE \cdot Eh$;
conseq. the square $AL^2 =$ the rect. $HE \cdot Eh$.
But, by sim. tri. $PA : AL :: GE : EH$,
and, by the same, $QA : Al :: EK : Eh$;
theref. by comp. $PA \cdot AQ : AL^2 :: GE \cdot EK : HE \cdot Eh$;
and because $AL^2 = HE \cdot Eh$, theref. $PA \cdot AQ = GE \cdot EK$.

But the parallelograms CGEK, CPAQ, being equiangular, are as the rectangles $GE \cdot EK$ and $PA \cdot AQ$.

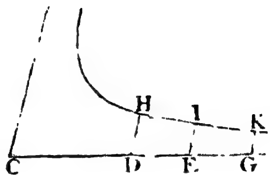
Therefore the parallelogram GK = the paral. PQ.

That is, all the inscribed parallelograms are equal to one another. Q. E. D.

Corol. 1. Because the rectangle GEK or CGE is constant, therefore GE is reciprocally as CG, or CG : CP :: PA : GE. And hence the asymptote continually approaches towards the curve, but never meets it: for GE decreases continually

as CG increases; and it is always of *some* magnitude, except when CG is supposed to be infinitely great, for then GE is infinitely small, or nothing. So that the asymptote CG may be considered as a tangent to the curve at a point infinitely distant from C.

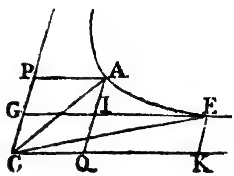
Corol. 2. If the abscisses CD, CE, CG, &c., taken on the one asymptote, be in geometrical progression increasing; then shall the ordinates DH, EI, GK, &c. parallel to the other asymptote, be a decreasing geometrical progression, having the same ratio. For, all the rectangles CDH, CEI, CGK, &c. being equal, the ordinates DH, EI, GK, &c. are reciprocally as the abscisses CD, CE, CG, &c. which are geometricals. And the reciprocals of geometricals are also geometricals, and in the same ratio, but decreasing, or in converse order.



THEOREM XIII.

The three following Spaces between the Asymptotes and the Curve, are equal; namely, the Sector or Trilinear Space contained by an Arc of the Curve and two Radii, or Lines drawn from its Extremities to the Centre; and each of the two Quadrilaterals, contained by the said Arc, and two Lines drawn from its Extremities parallel to one Asymptote, and the intercepted Part of the other Asymptote.

That is,
The sector CAE = PAEG = QAEK,
all standing on the same arc AE.



FOR, by theor. 12, CPAQ = CGEK;
subtract the common space CGIQ,
there remains the paral. PI = the par. IK;
To each add the trilineal IAE, then
the sum is the quadr. PAEG = QAEK.

Again, from the quadrilateral CAEK
take the equal triangles CAQ, CEK,
and there remains the sector CAE = QAEK.
Therefore CAE = QAEK = PAEG. Q. E. D.

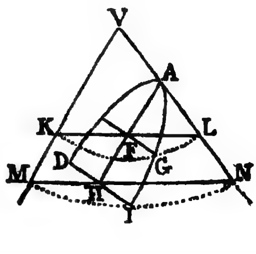
SCHOLIUM.

In the figure to theorem 12, cor. 1. if CD = 1, and CE, CG, &c. be any numbers, the hyperbolic spaces HDEI, IEGK, &c. are analogous to the *logarithms* of those numbers. For, whilst the numbers CD, CE, CG, &c. proceed in geometrical progression, the correspondent spaces proceed in arithmetical progression; and therefore, from the nature of logarithms are respectively proportional to the logarithms of those numbers. If the angle C were a right angle, and CD = DH = 1; then if CE were = 10, the space DEIH would be 2·30258509, &c.; if CG were = 100, then the space DGKH would be 4·60517018: these being the Napierean logarithms to 10 and 100 respectively. Intermediate areas corresponding to intermediate abscissæ would be the appropriate logarithms. These are usually called *Hyperbolic* logarithms; but the term is improper: for by drawing other hyperbolic curves between HIK and its asymptotes, other systems of logarithms would be obtained. Or, by changing the angle between the asymptotes the same thing may be effected. Thus, when the angle C is a right angle, or has its sine = 1, the hyperbolic spaces indicate the Napierean logarithms; but when the angle is $25^{\circ}\ 44'\ 27\frac{1}{4}''$, whose sine is = ·43429448, &c. the modulus to the common, or Briggs's, logarithms, the spaces DEIH, &c. measure those logarithms. In both cases, if spaces to the right of DH are regarded as *positive*, those to the left will be *negative;* whence it follows that the logarithms of numbers less than 1 are negative also.

OF THE PARABOLA.

THEOREM I.

The Abscisses are proportional to the Squares of their Ordinates.



Let AVM be a section through the axis of the cone, and AGIH a parabolic section by a plane perpendicular to the former, and parallel to the side VM of the cone; also let AFH be the common intersection of the two planes, or the axis of the parabola, and FG, HI ordinates perpendicular to it.

Then it will be, as AF : AH :: $FG^2 : HI^2$.

For, through the ordinates FG, HI draw the circular sections, KGL, MIN, parallel to the base of the cone, having KL, MN for their diameters, to which FG, HI are ordinates, as well as to the axis of the parabola.

Then, by similar triangles, AF : AH :: FL : HN;
but, because of the parallels, KF = MH;
therefore - - - AF : AH :: KF . FL : MH . HN.
But, by the circle, KF . FL = FG^2, and MH . HN = HI^2;
Therefore - - - AF : AH :: $FG^2 : HI^2$. Q. E. D.

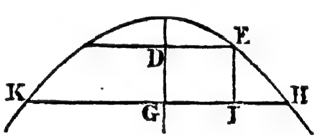
Corol. Hence the third proportional $\frac{FG^2}{AF}$ or $\frac{HI^2}{AH}$ is a constant quantity, and is equal to the parameter of the axis by defin. 16.

Or AF : FG :: FG : P the parameter.
Or the rectangle P . AF = FG^2.

THEOREM II.

As the Parameter of the Axis:
Is to the Sum of any Two Ordinates::
So is the Difference of those Ordinates:
To the Difference of their Abscisses.

That is,
P : GH + DE :: GH − DE : DG,
Or, P : KI :: IH : IE.



For, by cor. theor. 1, P . AG = GH^2,
and - - - P . AD = DE^2;
theref. by subtraction, P . DG = $GH^2 - DE^2$.
Or, - - - P . DG = KI . IH,
therefore - - P : KI :: IH : DG or EI. Q. E. D.

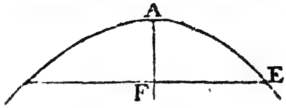
Corol. Hence, because P . EI = KI . IH,
and, by cor. theor. 1, P . AG = GH^2,
therefore - - AG : EI :: GH^2 : KI . IH.

So that any diameter EI is as the rectangle of the segments KI, IH of the double ordinate KH.

THEOREM III.

The Distance from the Vertex to the Focus is equal to $\frac{1}{4}$ of the Parameter, or to Half the Ordinate at the Focus.

That is,
$AF = \frac{1}{2}FE = \frac{1}{4}P$,
where F is the focus.

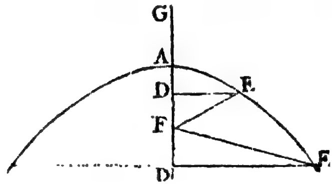


For, the general property is $AF : FE :: FE \quad P$.
But, by definition 17, - $FE = \frac{1}{2}P$;
therefore also - - $AF = \frac{1}{2}FE = \frac{1}{4}P$. Q. E. D.

THEOREM IV.

A Line drawn from the Focus to any Point in the Curve, is equal to the Sum of the Focal Distance and the Absciss of the Ordinate to that Point.

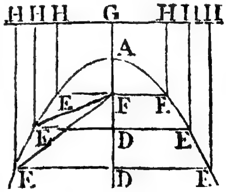
That is,
$FE = FA + AD = GD$,
taking $AG = AF$.



For, since $FD = AD \sim AF$,
theref. by squaring, $FD^2 = AF^2 - 2AF \cdot AD + AD^2$,
But, by cor. theor. 1, $DE^2 = P \cdot AD = 4AF \cdot AD$;
theref. by addition, $FD^2 + DE^2 = AF^2 + 2AF \cdot AD + AD^2$.
But, by right-ang. tri. $FD^2 + DE^2 = FE^2$;
therefore - - $FE^2 = AF^2 + 2AF \cdot AD + AD^2$,
and the root or side is $FE = AF + AD$,
or - - - $FE = GD$, by taking $AG = AF$.

Q. E. D.

Corol. 1. If, through the point G, the line GH be drawn perpendicular to the axis, it is called the directrix of the parabola.* The property of which, from this theorem, it appears, is this: That drawing any lines HE parallel to the axis, HE is always equal to FE the distance of the focus from the point E.



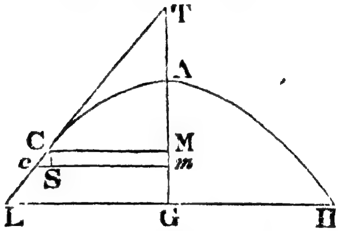
* Each of the other conic sections has a directrix; but the consideration of it does not occur in the mode here employed of investigating the general properties of the curves.

Corol. 2. Hence also the curve is easily described by points. Namely, in the axis produced take AG = AF the focal distance, and draw a number of lines EE perpendicular to the axis AD; then with the distances GD, GD, GD, &c. as radii, and the centre F, draw arcs crossing the parallel ordinates in E, E, E, &c. Then draw the curve through all the points E, E, E.

THEOREM V.

If a Tangent be drawn to any Point of the Parabola, meeting the Axis produced; and if an Ordinate to the Axis be drawn from the Point of Contact; then the Absciss of that Ordinate will be equal to the External Part of the Axis, measured from the Vertex.

That is, if TC touch the curve at the point C, then is AT = AM.



Let *c*C, an indefinitely small portion of a parabolic curve, be produced to meet the prolongation of the axis in T; and let *cm* be drawn parallel to CM, and *c*S parallel to AG the axis. Let, also, p = parameter of the parabola.

Then, by sim. tri. $cs : sc :: CM : MA + AT = MT$,

$$\therefore cs = \frac{MT \cdot cs}{CM}.$$

Also, th. 1. cor. $p \cdot Am = mc^2 = ms^2 + 2ms \cdot sc + sc^2,$

$$= MC^2 + 2MC \cdot sc + sc^2,$$

$$\text{and } p \cdot AM = MC^2.$$

Consequently, omitting sc^2 as indefinitely small, and subtracting the latter equa. from the former, we have

$$p \cdot (Am - AM) = p \cdot cs = 2cs \cdot MC:$$

or, substituting for cs its value above,

$$p \cdot \frac{MT \cdot cs}{CM} = 2cs \cdot MC;$$

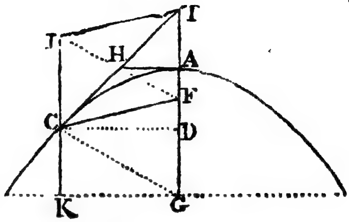
or $p \cdot MT = 2MC^2 = 2p \cdot AM$ (th. 1.)

Consequently, $MT = 2AM$, and $MA = AT$. Q. E. D.

THEOREM VI.

If a Tangent to the Curve meet the Axis produced; then the Line drawn from the Focus to the Point of Contact, will be equal to the Distance of the Focus from the Intersection of the Tangent and Axis.

That is, $FC = FT$.



For, draw the ordinate DC to the point of contact C.

Then, by theor. 5, $AT = AD$;

therefore - $FT = AF + AD$.

But, by theor. 4, $FC = AF + AD$;

theref. by equality, $FC = FT$. Q. E. D.

Corol. 1. If CG be drawn perpendicular to the curve, or to the tangent, at C; then shall $FG = FC = FT$.

For, draw FH perpendicular to TC, which will also bisect TC, because $FT = FC$; and therefore, by the nature of the parallels, FH also bisects TG in F. And consequently $FG = FT = FC$.

So that F is the centre of a circle passing through T, C, G.

Corol. 2. The subnormal DG is a constant quantity, and equal to half the parameter, or to 2AF, double the focal distance. For, since TCG is a right angle,

therefore TD or $2AD : DC :: DC : DG$;

but by the def. $AD : DC :: DC :$ parameter;

therefore $DG =$ half the parameter $= 2AF$.

Corol. 3. The tangent at the vertex AH, is a mean proportional between AF and AD.

For, because FHT is a right angle,

therefore - AH is a mean between AF, AT,

or between - AF, AD, because $AD = AT$.

Likewise, - FH is a mean between FA, FT,

or between FA, FC.

Corol. 4. The tangent TC makes equal angles with FC and the axis FT; as well as with FC and CI.

For, because $FT = FC$,

therefore the $\angle FCT = \angle FTC$.

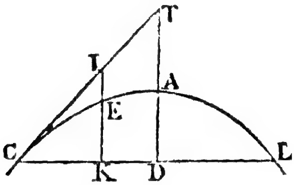
Also, the angle GCF = the angle GCK, drawing ICK parallel to the axis AG.

Corol. 5. And because the angle of incidence GCK is = the angle of reflection GCF; therefore a ray of light falling on the curve in the direction KC, will be reflected to the focus F. That is, all rays parallel to the axis, are reflected to the focus, or burning point.

THEOREM VII.

If there be any Tangent, and a Double Ordinate drawn from the Point of Contact, and also any Line parallel to the Axis, limited by the Tangent and Double Ordinate: then shall the Curve divide that Line in the same Ratio as the Line divides the Double Ordinate.

That is,
IE : EK :: CK : KL.

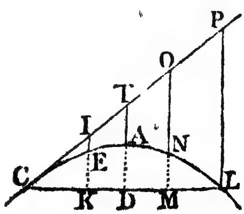


For, by sim. triangles, CK : KI :: CD : DT or 2DA
but, by the def. the param. P : CL :: CD : 2DA;
therefore, by equality, P : CK :: CL : KI.
But, by theor. 2, P : CK :: KL : KE;
therefore, by equality, CL : KL :: KI : KE;
and, by division, - CK : KL :: IE : EK. Q. E. D.

THEOREM VIII.

The same being supposed as in theor. 7; then shall the External Part of the Line between the Curve and Tangent, be proportional to the Square of the intercepted Part of the Tangent, or to the Square of the intercepted Part of the Double Ordinate.

That is, IE is as CI^2 or as CK^2,
and IE, TA, ON, PL, &c.
are as CI^2, CT^2, CO^2, CP^2, &c.
or as CK^2, CD^2, CM^2, CL^2, &c.



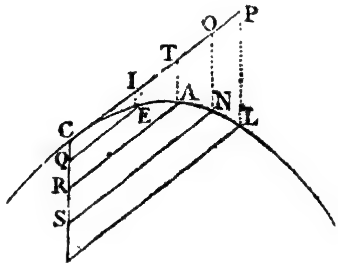
For, by theor. 7, IE : EK :: CK : KL,
or, by ———, IE : EK :: CK^2 : CK . KL.
But, by cor. th. 2, EK is as the rect. CK . KL,
therefore - IE is as CK^2, or as CI^2. Q. E. D.

Corol. As this property is common to every position of the tangent, if the lines IE, TA, ON, &c. be appended on the points I, T, O, &c. and moveable about them, and of such lengths as that their extremities E, A, N, &c. be in the curve of a parabola in some one position of the tangent; then making the tangent revolve about the point C, it appears that the extremities E, A, N, &c. will always form the curve of some parabola, in every position of the tangent.

THEOREM IX.

The Abscisses of any Diameter, are as the Squares of their Ordinates.

That is, CQ, CR, CS, &c.
are as QE^2, RA^2, SN^2, &c.
Or CQ : CR :: QE^2 : RA^2, &c.



For, draw the tangent CT, and the externals EI, AT, NO, &c, parallel to the axis, or to the diameter CS.

Then, because the ordinates QE, RA, SN, &c, are parallel to the tangent CT, by the definition of them, therefore all the figures IQ, TR, OS, &c, are parallelograms, whose opposite sides are equal;

namely, - - IE, TA, ON, &c,
are equal to - CQ, CR, CS, &c,
Therefore, by theor. 8, CQ, CR, CS, &c,
are as - - CI^2, CT^2, CO^2, &c.
or as their equals - QE^2, RA^2, SN^2, &c. Q. E. D.

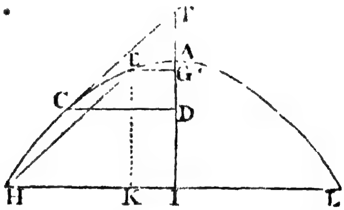
Corol. Here, like as in theor. 2, the difference of the abscisses is as the difference of the squares of their ordinates, or as the rectangles under the sum and difference of the ordinates, the rectangle of the sum and difference of the

ordinates being equal to the rectangle under the difference of the abscisses and the parameter of that diameter, or a third proportional to any absciss and its ordinate.

THEOREM X.

If a Line be drawn parallel to any Tangent, and cut the Curve in two Points; then, if two Ordinates be drawn to the Intersections, and a third to the Point of Contact, these three Ordinates will be in Arithmetical Progression, or the Sum of the Extremes will be equal to Double the Mean.

That is,
EG + HI = 2CD.



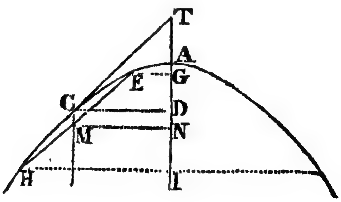
For, draw EK parallel to the axis, and produce HI to L.
Then, by sim. triangles, EK : HK :: TD or 2AD : CD;
but, by theor. 2, - EK : HK : KL : P the param.
theref. by equality, 2AD : KL CD : P.
But, by the defin. 2AD : 2CD :: CD : P;
theref. the 2d terms are equal, KL = 2CD,
that is, - - EG + HI = 2CD. Q. E. D.

Corol. When the point E is on the other side of AI; then HI − GE 2CD.

THEOREM XI.

Any diameter bisects all its Double Ordinates, or Lines parallel to the Tangent at its Vertex.

That is,
ME = MH.



For, to the axis AI draw the ordinates EG, CD, HI, and MN parallel to them, which is equal to CD.

Then, by theor. 10, $2MN$ or $2CD = EG + HI$, therefore M is the middle of EH.

And, for the same reason, all its parallels are bisected.

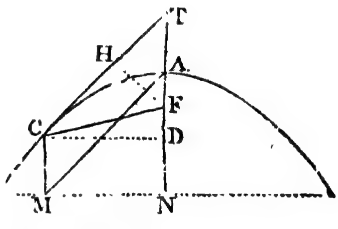
Q. E. D.

SCHOL. Hence, as the abscisses of any diameter and their ordinates have the same relations as those of the axis, namely, that the ordinates are bisected by the diameter, and their squares proportional to the abscisses; so all the other properties of the axis and its ordinates and abscisses, before demonstrated, will likewise hold good for any diameter and its ordinates and abscisses. And also those of the parameters, understanding the parameter of any diameter, as a third proportional to any absciss and its ordinate. Some of the most material of which are demonstrated in the four following theorems.

THEOREM XII.

The Parameter of any Diameter is equal to four Times the Line drawn from the Focus to the Vertex of that Diameter.

That is, $4FC = p$, the param. of the diam. CM.



For, draw the ordinate MA parallel to the tangent CT: also CD, MN perpendicular to the axis AN, and FH perpendicular to the tangent CT.

Then the abscisses AD, CM or AT, being equal, by theor. 5, the parameters will be as the squares of the ordinates CD, MA or CT, by the definition;

that is, - - $P : p :: CD^2 : CT^2$,
But, by sim. tri. - $FH : FT :: CD : CT$;
therefore - - $P : p :: FH^2 \ FT^2$.
But, by cor. 2, th. 6, $FH^2 = FA \cdot FT$;
therefore - - $P : p :: FA \cdot FT : FT^2$;
or, by equality, - $P : p :: FA : FT$ or FC.

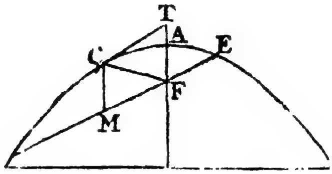
But, by theor. 3, $P = 4FA$,
and therefore - $p = 4FT$ or $4FC$. Q. E. D.

Corol. Hence the parameter p of the diameter CM is equal to $4FA + 4AD$, or to $P + 4AD$, that is, the parameter of the axis added to 4AD.

THEOREM XIII.

If an Ordinate to any Diameter pass through the Focus, it will be equal to Half its Parameter; and its Absciss equal to One Fourth of the same Parameter.

That is, $CM = \frac{1}{4}p$,
and $ME = \frac{1}{2}p$.



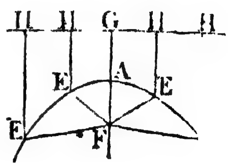
For, join FC, and draw the tangent CT.

By the parallels, $CM = FT$;
and, by theor. 6, $FC = FT$;
also, by theor. 12, $FC = \frac{1}{4}p$;
therefore - - $CM = \frac{1}{4}p$.

Again, by the defin. CM or $\frac{1}{4}p$: ME : : ME : p,
and consequently $ME = \frac{1}{2}p = 2CM$. Q. E. D.

Corol. 1. Hence, of any diameter, the double ordinate which passes through the focus, is equal to the parameter, or to quadruple its absciss.

Corol. 2. Hence, and from cor. 1. to theor. 4, and theor. 6 and 12, it appears, that if the directrix GH be drawn, and any lines HE, HE, parallel to the axis; then every parallel HE will be equal to EF, or $\frac{1}{4}$ of the parameter of the diameter to the point E.

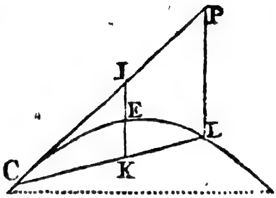


THEOREM XIV.

If there be a tangent, and any Line drawn from the Point of Contact and meeting the Curve in some other Point, as also another Line parallel to the Axis, and limited by the First Line and the Tangent: then shall the Curve divide this Second Line in the same Ratio, as the Second Line divides the First Line.

That is,

IE : EK : : CK : KL.



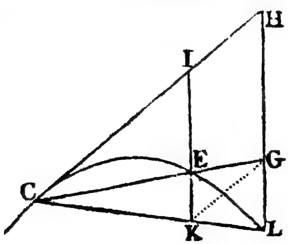
For, draw LP parallel to IK, or to the axis.

Then by theor. 8,	$IE : PL :: CI^2 : CP^2$,
or, by sim. tri. -	$IE : PL :: CK^2 : CL^2$.
Also, by sim. tri.	$IK : PL :: CK : CL$,
or - - -	$IK : PL :: CK^2 : CK . CL$;
therefore by equality,	$IE : IK :: CK . CL : CL^2$;
or - - -	$IE : IK :: CK : CL$;
and, by division,	$IE : EK :: CK : KL$. Q. E. D.

Corol. When $CK = KL$, then $IE = EK = \frac{1}{2}IK$.

THEOREM XV.

If from any Point of the Curve there be drawn a Tangent, and also Two Right Lines to cut the Curve; and Diameters be drawn through the Points of Intersection E and L, meeting those Two Right Lines in two other Points G and K: Then will the Line KG joining these last Two Points be parallel to the Tangent.

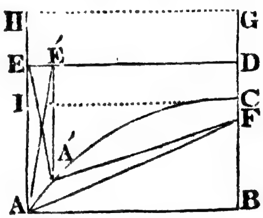


For, by theor. 14, CK : KL : : EI : EK;
and by composition, CK : CL : : EI : KI;
and by the parallels CK : CL : : GH : LH;
But, by sim. tri. - CK : CL : : KI : LH;
theref. by equal. KI : LH : : GH : LH :
consequently - KI = GH,
and therefore - KG is parallel and equal to IH. Q. E. D.

THEOREM XVI.

The Area or Space of a Parabola, is equal to Two-Thirds of its Circumscribing Parallelogram.

Let ACB be a semi-parabola, CB the axis, F the focus, ED the directrix; then if the line AF be supposed to revolve about F as a centre, while the line AE moves along the directrix perpendicularly to it, the area generated by the motion of AE, will always be equal to double the area generated by FA; and consequently the whole external area AEGD = double the area ACF.



For draw A'E' parallel, and indefinitely near, to AE; and draw the diagonals AE' and A'E; then by th. 6, cor. 4. the angles E'A'A and FA'A are equal, AA' being considered as part of the tangent at A'; and in the same manner, the angles EAA' and FAA' are also equal to each other; and since EA = AF, and E'A' = A'F; the triangles EAA' and E'A'A are each equal to the triangle AA'F; but the triangle EAA' = the triangle EE'A, being on the same base and between the same parallels; therefore the sum of the two triangles EE'A and EA'A, or the quadrilateral space EAA'E' is double the trilateral space AA'F; and as this is the case in every position of FA', E'A', it follows that the whole external area EACD = double the internal area AFC.

Hence, Take DG = FB, and complete the parallelogram DGHE, which is double the triangle ABF; therefore the area ABC = $\frac{1}{2}$ the area HACG, or $\frac{1}{2}$ of the rectangle ABGH, or $\frac{2}{3}$ of the rectangle ABCI, because BC = $\frac{1}{2}$ BG; that is the area of a parabola = $\frac{2}{3}$ of the circumscribing rectangle. Q. E. D*.

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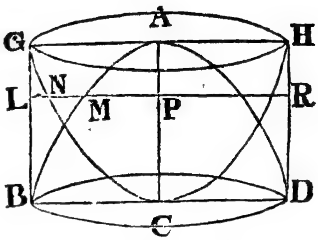
* This demonstration was given by Lieut. Drummond of the Royal Engineers, when he was a gentleman Cadet at the Royal Military Academy.

THEOREM XVII.

The Solid Content of a Paraboloid (or Solid generated by the Rotation of a Parabola about its Axis), is equal to Half its Circumscribing Cylinder.



Let GHBD be a cylinder, in which two equal paraboloids are inscribed; one BAD having its base BCD equal to the lower extremity of the cylinder; the other GCH inverted with respect to the former, but of equal base and altitude. Let the plane LR parallel to each end of the cylinder, cut all the three solids, while a vertical plane may be supposed to cut them so as to define the parabolas shown in the figure.

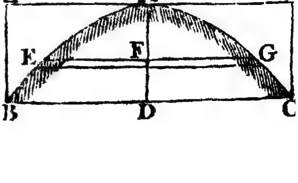
Then, in the semi-parabola ACB, $p \cdot AP = PM^2$,
also, in the semi-parabola ACG, $p \cdot CP = PN^2$,
consequently, by addition, $p.(AP + CP) = p \cdot AC = PM^2 + PN^2$.
But, $p \cdot AC = CB^2 = PL^2$.
Therefore $PL^2 = PM^2 + PN^2$:
That is, since circles are as the squares of their radii, the circular section of the cylinder, is equal to the sum of the corresponding sections of the two paraboloids.

The same property evidently obtains for any sections whatever parallel to BD; it therefore holds for the two paraboloids. In other words, the cylinder is equal to the two paraboloids taken together: wherefore, since the two paraboloids, having equal bases and equal altitudes, are equal to one another, it follows that each paraboloid is half of its circumscribing cylinder. Q. E. D.

THEOREM XVIII.

The Solidity of the Frustum BEGC of the Paraboloid, is equal to a Cylinder whose Height is DF, and its Base Half the Sum of the two Circular Bases EG, BC.

Let $c = 3{\cdot}1416$:



Then, by the last theor. $\frac{1}{2}pc \times AD^2 =$ the solid ABC,
and, by the same $\frac{1}{2}pc \times AF^2 =$ the solid AEG,
theref. the diff. $\frac{1}{2}pc \times (AD^2 - AF^2) =$ the frust. BEGC.
But $AD^2 - AF^2 = DF \times (AD + AF)$,
theref. $\frac{1}{2}pc \times DF \times (AD + AF) =$ the frust. BEGC.
But, by th. 1. $p \times AD = DC^2$, and $p \times AF = FG^2$;
theref. $\frac{1}{2}c \times DF \times (DC^2 + FG^2) =$ the frust. BEGC.

Q. E. D.

PROBLEMS, &c. FOR EXERCISE IN CONIC SECTIONS.

1. Demonstrate that if a cylinder be cut obliquely the section will be an ellipse.

2. Show how to draw a tangent to an ellipse whose foci are F, f, from a given point P.

3. Show how to draw a tangent to a given parabola from a given point P.

4. The diameters of an ellipse are 16 and 12. Required the parameter and the area.

5. The base and altitude of a parabola are 12 and 9. Required the parameter, and the semi-ordinates corresponding to the abscissæ 2, 3, and 4.

6. In the actual formation of arches, the voussoirs or arch-stones are so cut as to have their faces always *perpendicular* to the respective points of the curve upon which they stand. By what constructions may this be effected for the parabola and the ellipse?

7. Construct accurately on paper, a parabola whose base shall be 12 and altitude 9.

8. A cone, the diameter of whose base is 10 inches, and whose altitude is 12, is cut obliquely by a plane, which enters at 3 inches from the vertex on one slant side, and comes out at 3 inches from the base on the opposite slant side. Required the dimensions of the section?

9. Suppose the same cone to be cut by a plane parallel to one of the slant sides, entering the other slant side at 4 inches from the vertex, what will be the dimensions of the section?

10. Let any straight line EFR be drawn through F, one of the foci, of an ellipse, and terminated by the curve in E and R; then it is to be demonstrated that EF.FR = ER.$\frac{1}{4}$ parameter.

11. Demonstrate that, in any conic section, a straight line drawn from a focus to the intersection of two tangents makes equal angles with straight lines drawn from the same focus to the points of contact.

12. In every conic section the radius of curvature at any point is to half the parameter, in the triplicate ratio of the distance of the focus from that point to its distance from the tangent.

Also, in every conic section the radius of curvature is proportional to the cube of the normal.

Also, let PC be the radius of curvature at any point, P, in an ellipse or hyperbola whose transverse axis is AB, conjugate ab, and foci F and f: then is $\text{PC} = \frac{(\text{PF} \cdot \text{P}f)^{\frac{3}{2}}}{\frac{1}{4}\text{AB} \cdot ab}$.

Required demonstrations of these properties.

MECHANICS.

Definitions and preliminary Notions.

1. *Mechanics* is the science of equilibrium and of motion.

2. Every cause which moves, or tends to move a body, is called a *force.*

3. When the forces that are applied simultaneously to a body, destroy or annihilate each other's effects, then there is *equilibrium.*

4. *Statics* has for its object the equilibrium of forces applied to *solid* bodies.

5. By *Dynamics* we investigate the circumstances of the motion of solid bodies.

6. *Hydrostatics* is the science in which the equilibrium of fluids is considered.

7. *Hydrodynamics* is that in which the circumstances of their motion is investigated.

According to this division, *Pneumatics,* which relates to the properties of *elastic fluids,* is a branch of *Hydrostatics.*

For farther elucidation the following definitions, also, may advantageously find a place here, viz.

8. Body is the mass, or quantity of matter, in any mate-

rial substance; and it is always proportional to its weight or gravity, whatever its figure may be.

Body is either Hard, Soft, or Elastic. A Hard Body is that whose parts do not yield to any stroke or percussion, but retains its figure unaltered. A Soft Body is that whose parts yield to any stroke or impression, without restoring themselves again; the figure of the body remaining altered. And an Elastic Body is that whose parts yield to any stroke, but which presently restore themselves again, and the body regains the same figure as before the stroke.

We know of no bodies that are absolutely, or perfectly, either hard, soft, or elastic; but all partaking these properties, more or less, in some intermediate degree.

9. Bodies are also either Solid or Fluid. A Solid Body is that whose parts are not easily moved among one another, and which retains any figure given to it. But a Fluid Body is that whose parts yield to the slightest impression, being easily moved among one another; and its surface, when left to itself, is always observed to settle in a smooth plane at the top.

10. Density is the proportional weight or quantity of matter in any body. So, in two spheres, or cubes, &c. of equal size or magnitude; if the one weigh only one pound, but the other two pounds; then the density of the latter is double the density of the former; if it weigh three pounds, its density is triple; and so on.

11. Motion is a continual and successive change of place. —If the body move equally, or pass over equal spaces in equal times, it is called Equable or Uniform Motion. But if it increase or decrease, it is Variable Motion; and it is called Accelerated Motion in the former case, and Retarded Motion in the latter.—Also, when the moving body is considered with respect to some other body at rest, it is said to be Absolute Motion. But when compared with others in motion, it is called Relative Motion.

12. Velocity, or Celerity, is an affection of motion, by which a body passes over a certain space in a certain time. Thus, if a body in motion pass uniformly over 40 feet in 4 seconds of time, it is said to move with the velocity of 10 feet per second; and so on.

13. Momentum, or Quantity of Motion, is the power or force in moving bodies, by which they continually tend from their present places, or with which they strike any obstacle that opposes their motion.

14. Forces are distinguished into Motive, and Accelerative or Retarding. A Motive or Moving Force, is the power of an agent to produce motion; and it is equal or proportional to the momentum it will generate in any body, when acting, either by percussion, or for a certain time as a permanent force.

15. Accelerative, or Retardive Force, is commonly understood to be that which affects the velocity only: or it is that by which the velocity is accelerated or retarded; and it is equal or proportional to the motive force directly, and to the mass or body moved inversely.—So, if a body of 2 pounds weight, be acted on by a motive force of 40; then the accelerating force is 20. But if the same force of 40 act on another body of 4 pounds weight; then the accelerating force in this latter case is only 10; and so is but half the former, and will produce only half the velocity.

16. Gravity, or Weight, is that force by which a body endeavours to fall downwards. It is called Absolute Gravity, when the body is in empty space; and Relative Gravity, when immersed in a fluid.

17. Specific Gravity is the relation of the weights of different bodies of equal magnitude; and so is proportional to the density of the body.

NEWTONIAN AXIOMS.

18. Every body naturally endeavours to continue in its present state, whether it be at rest, or moving uniformly in a right line.

19. The change or Alteration of Motion, by any external force, is always proportional to that force, and in the direction of the right line in which it acts.

20. Action and Re-action, between any two bodies, are equal and contrary. That is, by Action and Re-action, equal changes of motion are produced in bodies acting on each other; and these changes are directed towards opposite or contrary parts.

STATICS.

21. The relative magnitudes and directions of any two forces may be represented by two right lines, which shall bear to each other the relations of the forces, and which shall

be inclined to each other in an angle equal to that made by the directions of the forces.

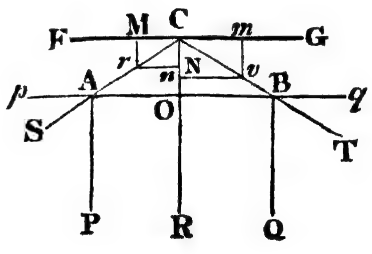
22. The name *resultant* is given to a force which is equivalent to two or more forces acting at once upon a point, or upon a body; these separate forces being named *constituents* or *composants*.

23. The operation by which the *resultant* of two or more forces applied to the same point, or line, or body, is determined, is called the *composition of forces;* the inverse problem is called the *decomposition,* or the *resolution of forces.*

24. The resultant of two or more forces which act upon the same line, in the same direction, is equal to their sum; and if some forces act in one direction, and others in a direction immediately opposite, the resultant will be equal to the excess of the sum of the forces which act in one direction above the sum of those which act in the opposite direction.

Composition and Resolution of Parallel Forces.

25. PROP. If to the extremities of an inflexible right line AB, are applied two forces, P and Q, whose directions are parallel and whose actions concur:—1st, The direction of the resultant, R, of those two forces is parallel to the right lines AP, BQ, and is equal to their sum. 2dly, That resultant divides the line AB into two parts reciprocally proportional to the two forces.



1. It is manifest that if two new forces, p and q, equal and in opposite directions are applied to the line AB, they will make no change in the state of the system; so that the resultant of the four forces p, q, P, Q, will be the same as that of the resultant of the two original forces P, Q. Suppose, now, that S is the resultant of the two forces p, P, while T is that of the two forces q, Q. These resultants, lying in the same place, will, if prolonged, necessarily meet in some point C; to which, therefore, we may suppose the forces S and T applied.

Through this point let FG be drawn parallel to AB, and suppose each of the forces S and T resolved into two forces

directed respectively in FG and CR. The forces, according to FG, being equal to p and q respectively, and applied in opposite directions, destroy each others' effects: the remaining forces, therefore, lying the same way on CR must be added together for the resultant, which thus is equal to P + Q; being the first part of the proposition.

2. In order to establish the second part of the proposition, let MC, CN, be lines in proportion to each other as the forces p, P; and in C, Cn, respectively proportional as q, Q: and draw Nr, nv, parallel to AB.

Then, by the sim. triangles CNr, COA; cnv, COB;

$$\left.\begin{array}{l} P : p :: CN : Nr :: CO : OA \\ q : Q :: nv : nc :: OB : OC. \end{array}\right.$$

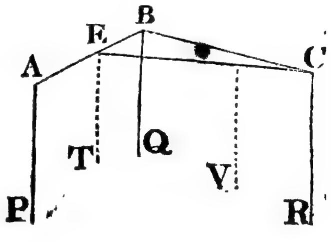
Consequently, $P \,.\, q : p \,.\, Q :: CO \,.\, OB : CO \,.\, OA$,

or, since $p = q$, it is $P : Q :: OB : OA$. Q. E. D.

Corol. 1. If $P = Q$, $BO = OA$.

Corol. 2. When a single force R is applied to a point O, of an inflexible straight line AB, we may always resolve it, or conceive it resolved, into two others, which being applied to the two points A and B, in directions parallel to R, shall produce the same effect.

26. PROP. Any number of parallel forces, P, Q, R, S, &c. acting in the same sense, and their points of application being connected in an invariable manner; to determine their resultant.



Determining first, by the preceding prop. the resultant T of two of the forces P and Q, we shall have $T = P + Q$;

$$P + Q : Q :: AB : AE.$$

Thus, we may substitute for the forces P and Q, the single force T whose value and point of application are known. Draw EC from that point of application to the point C, at which another force, R, is applied. Compounding the forces T and R, their resultant V will be $= T + R = P + Q + R$; and its point of application, F, such that

$$P + Q + R : R :: EC : EF.$$

A similar method may, obviously, be pursued for any number of parallel forces.

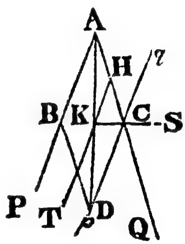
27. If parallel forces act in opposite directions; some, for example, upwards, others downwards; find the resultants of the first and of the second class separately, the general

resultant will be expressed by the difference of the two former.

28. The point through which the resultant of parallel forces passes, is called *the centre of parallel forces.* If the forces, without ceasing to be respectively parallel, and without changing either their magnitudes or their points of application, assume another general direction, the centre of those forces will still be the same, because the magnitudes and relations, on which its *position* depends, remain the same.

Concurring Forces.

29. Prop. The resultant of two forces P and Q acting in one plane, will be represented in direction and in magnitude, by the diagonal of the parallelogram constructed on the directions of those forces.



1. *In direction.* Take, on the directions AP, AQ, of the forces, P, Q, distances AB, AC, proportional to those forces, respectively. Suppose that the force Q is applied at the point C, and that at the same point two other forces p, q, equal to each other, act *in opposite directions*, each of those forces being, also, equal to Q.

The effect of the four forces P, Q, p, q, will evidently be the same as that of the primitive forces P, Q; since the other two annihilate each other's effects.

The forces Q, q, will have a resultant S, whose direction, CS, will bisect the angle, QCq, made by the direction of the other two; since no reason can be assigned why it should lean to one rather than toward the other.

The forces P, p, acting in parallel directions, would have a resultant, T, whose direction TH (art. 25.) would be parallel to them, and pass through a point, H, such as that P : p :: HC : HA.

Now, the point K, where the directions CS, TH, of these two resultants intersect, will evidently be a point in the direction of the resultant of the *four* forces P, p, Q, q; and, consequently, of the original forces P, Q.

But the triangle CHK is isosceles: for, since HT, CP, are parallel, the alternate angles DCK, HKC, are equal, and DCK, HCK, are equal, because SC bisects the angle QCq: hence, HCK = HKC, and HK = HC.

But, from what has preceded, P : Q :: HC : HA; and therefore P : p or Q :: HK : HA.

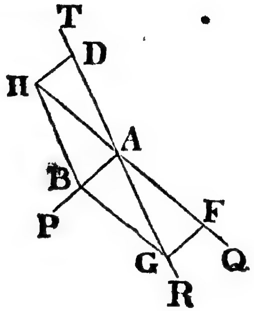
From B drawing BD parallel to AC, we shall have

P : Q :: AB : AC :: CD : AC,

whence CD : AC :: HK : HA;

a proportion which indicates that the three points, A, K, D, all fall on the diagonal of a parallelogram ABCD.

2. *In magnitude.* For, with regard to the forces P, Q, represented in magnitude and direction by AB and AF, let T be opposed to those two forces so as to keep the whole system in equilibrio: then it will, of necessity, be equal and opposite to their resultant, R, whose direction is AG. Now, if we suppose that the force Q is in equilibrio with the two forces P and T (which is consistent with our first hypothesis) the resultant of these latter will fall in the prolongation of QA, and will be represented by AH = AF. Also, if HD be drawn parallel to AB, and HB be joined, it will be equal and parallel to AG; and we shall have

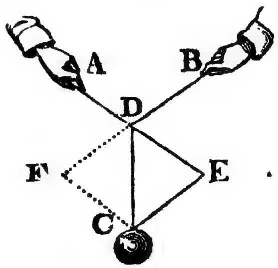


P : T :: AB : AD.

Consequently, since AB represents, or measures, the force P, AD will represent or measure the force T; and as that force is in equilibrio with the two forces P and Q, or with their resultant, R, this latter will be represented or measured by AG = AD; that is, by the diagonal of the parallelogram ABGF. Q. E. D.

30. *Corol.* 1. If three forces, as A, B, C, acting simultaneously in the same plane, keep one another in equilibrio, they will be respectively proportional to the three sides, DE, EC, CD, of a triangle which are drawn parallel to the directions of the forces AD, DB, CD.

For, producing AD, BD, and drawing CF, CE, parallel to them, then the force in CD is equivalent to the two AD, BD, by the supposition; but the force CD is also equivalent to the two ED and CE or FD; therefore, if CD represent the force C, then ED will represent its opposite force A, and CE, or FD,

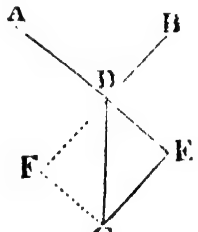


its opposite force B. Consequently the three forces A, B, C, are proportional to DE, CE, CD, the three lines parallel to the directions in which they act.

31. *Corol.* 2. Because the three sides CD, CE, DE, are proportional to the sines of their opposite angles E, D, C; therefore the three forces, when in equilibrio, are proportional to the sines of the angles of the triangle made of their lines of direction; namely, each force proportional to the sine of the angle made by the directions of the other two.

32. *Corol.* 3. The three forces, acting against, and keeping one another in equilibrio, are also proportional to the sides of any other triangle made by drawing lines either perpendicular to the directions of the forces, or forming any given angle with those directions. For such a triangle is always similar to the former, which is made by drawing lines parallel to the directions; and therefore their sides are in the same proportion to one another.

33. *Corol.* 4. If any number of forces be kept in equilibrio by their actions against one another; they may be all reduced to two equal and opposite ones.—For, any two of the forces may be reduced to one force acting in the same plane; then this last force and another may likewise be reduced to another force acting in their plane: and so on, till at last they be all reduced to the action of only two opposite forces; which will be equal, as well as opposite, because the whole are in equilibrio by the supposition.



34. *Corol.* 5. If one of the forces, as C, be a weight, which is sustained by two strings drawing in the directions DA, DB: then the force or tension of the string AD, is to the weight C, or tension of the string DC, as DE to DC; and the force or tension of the other string BD, is to the weight C, or tension of CD, as CE to CD.

35. *Corol.* 6. Since in any triangle CDE we have, by the principles of trigonometry,

$$DC^2 = DE^2 + EC^2 \pm 2DE \,.\, EC \cos. DEC,$$

it follows, that if F, f, be two forces that act simultaneously in directions, which make an angle A, then we may find the magnitude of the resultant, R, by the equation

$$R = \sqrt{(F^2 + f^2 \pm 2Ff \cos. A)}.$$

36. *Remark.*—The properties, in this proposition and its

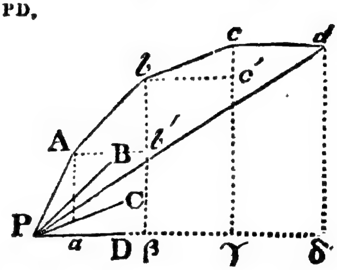
corollaries, hold true of all similar forces whatever, whether they be instantaneous or continual, or whether they act by percussion, drawing, pushing, pressing, or weighing; and are of the utmost importance in mechanics and the doctrine of forces.

37. If three forces, whose directions concur in one point, are represented by the three contiguous edges of a parallelopiped, their resultant will be represented, both in magnitude and direction, by the diagonal drawn from the point of concourse, to the opposite angle of the parallelopiped.

The demonstration of this is left for the exercise of the student.

38. PROP. To find the resultant of several forces concurring in one point, and acting in one plane.

1st. *Graphically.*—Let, for example, four forces, A, B, C, D, act upon the point P, in magnitudes and directions represented by the lines PA, PB, PC, PD.



From the point A draw Ab parallel and equal to PB; from b draw bc parallel and equal to PC; from c draw cd parallel and equal to PD; and so on, till all the forces have thus been brought into the construction. Then join Pd, which will represent both the magnitude and the direction of the required resultant.

This is, in effect, the same thing as finding the resultant of two of the forces A and B; then blending that resultant with a third force C; their resultant with a fourth force D; and so on.

2d. *By computation.* Drawing the lines Aa, Ab', &c. respectively parallel and perpendicular to the last force PD; we have

$$d\delta = \text{A}a + bb' + cc' = \text{A sin. APD} + \text{B sin. BPD} + \text{C sin. CPD}$$
$$\text{P}\delta = \text{P}\alpha + \alpha\beta + \beta\gamma + \gamma\delta = \text{A cos. APD} + \text{B cos. BPD} + \text{C cos. CPD} + \text{D}$$
$$\tan d\text{P}\delta = \frac{d\delta}{\text{P}\delta} \;\ldots\ldots\ldots\; \text{P}d = \sqrt{(\text{P}\delta^2 + d\delta^2)}.$$

The numerical computation is best effected by means of a table of natural sines.

39. *Remark.* Connected with this subject is the doctrine of *moments;* for an elucidation of which, however, the student should consult some of the books written expressly on mechanics, as those by *Marrat, Gregory*, or *Poisson.*

THE MECHANICAL POWERS, &c.

40. Weight and Power, when opposed to each other, signify the body to be moved, and the body that moves it; or the patient and agent. The power is the agent, which moves, or endeavours to move, the patient or weight.

41. Machine, or Engine, is any mechanical instrument contrived to move bodies. And it is composed of the mechanical powers.

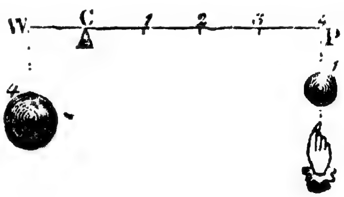
42. Mechanical Powers, are certain simple instruments, commonly employed for raising greater weights, or overcoming greater resistances, than could be effected by the natural strength without them. These are usually accounted six in number, viz. the Lever, the Wheel and Axle, the Pulley, the Inclined Plane, the Wedge, and the Screw.

43. Centre of Motion, is the fixed point about which a body moves. And the Axis of Motion, is the fixed line about which it moves.

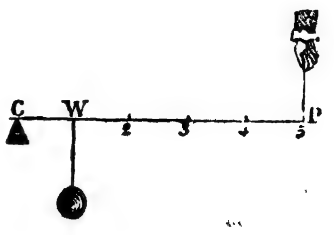
44. Centre of Gravity, is a certain point, on which a body being freely suspended, it will rest in any position.

OF THE LEVER.

45. A Lever is any inflexible rod, bar, or beam, which serves to raise weights, while it is supported at a point by a fulcrum or prop, which is the centre of motion. The lever is supposed to be void of gravity or weight, to render the demonstrations easier and simpler. There are three kinds of levers.

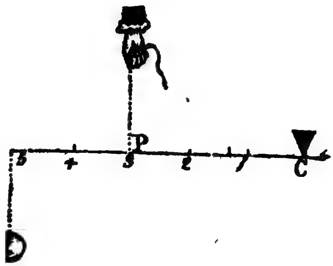


46. A Lever of the First kind has the prop c between the weight w and the power p. And of this kind are balances, scales, crows, hand-spikes, scissors, pinchers, &c.



47. A Lever of the Second kind has the weight between the power and the prop. Such as oars, rudders, cutting knives that are fixed at one end, &c.

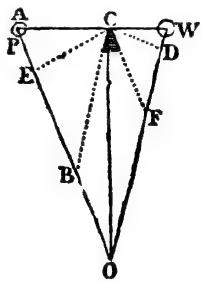
48. A Lever of the Third kind has the power between the weight and the prop. Such as tongs, the bones and muscles of animals, a man rearing a ladder, &c.



49. A Fourth kind is sometimes added, called the Bended Lever. As a hammer drawing a nail.

50. In all these instruments the power may be represented by a weight, which is its most natural measure, acting downward: but having its direction changed, when necessary, by means of a fixed pulley.

51. Prop. When the weight and power keep the lever in equilibrio, they are to each other reciprocally as the distances of their lines of direction from the prop. That is, P : W :: CD : CE; where CD and CE are perpendicular to WO and AO, the directions of the two weights, or the weight and power W and A.



For, draw CF parallel to AO, and CB parallel to WO: Also, join CO, which will be the direction of the pressure on the prop C; for there cannot be an equilibrium unless the directions of the three forces all meet in, or tend to, the same point, as O. Then, because these three forces keep each other in equilibrio, they are proportional to the sides of the triangle CBO or CFO, drawn in the direction of those forces; therefore - - - - - - P : W CF : FO or CB.
But, because of the parallels, the two triangles CDF, CEB are equiangular, therefore - - - CD : CE :: CF : CB.
Hence, by equality, - - P : W :: CD : CE.

That is, each force is reciprocally proportional to the distance of its direction from the fulcrum.

Another proof might easily be made out from art. 25, on parallel forces; but it will be found that this demonstration

will serve for all the other kinds of levers, by drawing the lines as directed.

52. *Corol.* 1. When the angle A is = the angle W, then is CD : CE : : CW : CA : : P : W. Or when the two forces act perpendicularly on the lever, as two weights, &c; then, in case of an equilibrium, D coincides with W, and E with P; consequently then the above proportion becomes also P : W : : CW : CA, or the distances of the two forces from the fulcrum, taken on the lever, are reciprocally proportional to those forces.

53. *Corol.* 2. If any force P be applied to a lever at A; its effect on the lever, to turn it about the centre of motion C, is as the length of the lever CA, and the sine of the angle of direction CAE. For the perp. CE is as CA × sin. ∠ A.

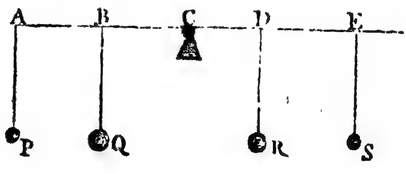
54. *Corol.* 3. Because the product of the extremes is equal to the product of the means, therefore the product of the power into the distance of its direction, is equal to the product of the weight into the distance of its direction.

That is, P × CE = W × CD.

55. *Corol.* 4. If the lever, with the weight and power fixed to it, be made to move about the centre C; the momentum of the power will be equal to the momentum of the weight; and their velocities will be in reciprocal proportion to each other. For the weight and power will describe circles whose radii are the distances CD, CE; and since the circumferences or spaces described are as the radii, and also as the velocities, therefore the velocities are as the radii CD, CE; and the momenta, which are as the masses and velocities, are as the masses and radii; that is, as P × CE and W × CD, which are equal by cor. 3.

56. *Corol.* 5. In a straight lever, kept in equilibrio by a weight and power acting perpendicularly; then, of these three, the power, weight, and pressure on the prop. any one is as the distance of the other two.

57. *Corol.* 6. If several weights P, Q, R, S, act on a straight lever, and keep it in equilibrio; then the sum of the products on one side of the prop. will be equal to the sum on the other side, made by multiplying each weight by its distance; namely,



(P × AC) + (Q × BC) = (R × DC) + (S × EC).

For, the effect of each weight to turn the lever, is as the weight multiplied into its distance; and in the case of an equilibrium, the sums of the effects, or of the products on both sides, are equal. The same would also follow from art. 26.

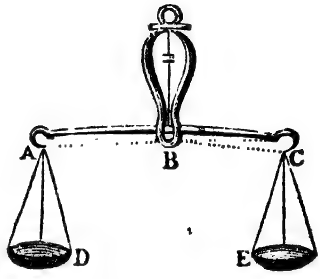
58. *Corol.* 7. Because, when two weights Q and R are in equilibrio, Q : R :: CD : CB;

therefore, by composition, Q + R : Q :: BD : CD,
and, Q + R : R :: BD : CB.

That is, the sum of the weights is to either of them, as the sum of their distances is to the distance of the other.

SCHOLIUM.

59. On the foregoing principles depends the nature of scales and beams, for weighing all sorts of goods. For, if the weights be equal, then will the distances be equal also, which gives the construction of the common scales, which ought to have these properties:



1*st*, That the points of suspension of the scales and the centre of motion of the beam, A, B, C, should be in a straight line: 2*d*, That the arms AB, BC, be of an equal length: 3*d*, That the centre of gravity be in the centre of motion B, or a little below it: 4*th*, That they be in equilibrio when empty: 5*th*, That there be as little friction as possible at the centre B. A defect in any of these properties, makes the scales either imperfect or false. But it often happens that the one side of the beam is made shorter than the other, and the defect covered by making that scale the heavier, by which means the scales hang in equilibrio when empty; but when they are charged with any weights, so as to be still in equilibrio, those weights are not equal; but the deceit will be detected by changing the weights to the contrary sides, for then the equilibrium will be immediately destroyed.

60. To find the true weight of any body by such a false balance:—First weigh the body in one scale, and afterwards weigh it in the other; then the mean proportional between these two weights, will be the true weight required. For, if any body *b* weigh W pounds or ounces in the scale D, and

only w pounds or ounces in the scale E: then we have these two equations, namely, $AB \cdot b = BC \cdot W$,

and $BC \cdot b = AB \cdot w$;

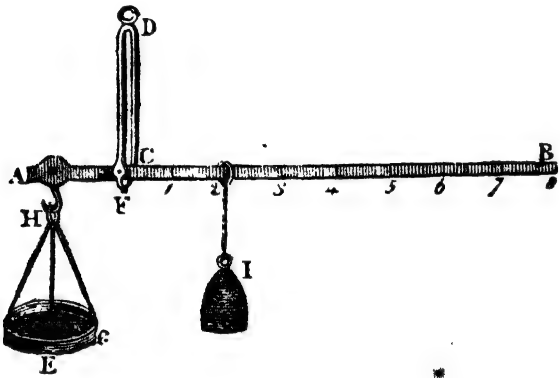
the product of the two is $AB \cdot BC \cdot b^2 = AB \cdot BC \cdot Ww$;

hence then - - - $b^2 = Ww$,

and - - - $b = \sqrt{Ww}$,

the mean proportional, which is the true weight of the body b.

61. The Roman Statera, or Steelyard, is also a lever, but of unequal brachia or arms, so contrived, that one weight only may serve to weigh a great many, by sliding it backward and forward, to different distances, on the longer arm of the lever; and it is thus constructed:

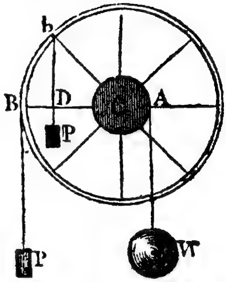


Let AB be the steelyard, and C its centre of motion, whence the divisions must commence if the two arms just balance each other: if not, slide the constant moveable weight I along from B towards C, till it just balance the other end without a weight, and there make a notch in the beam, marking it with a cipher 0. Then hang on at A a weight W equal to I, and slide I back towards B till they balance each other; there notch the beam, and mark it with 1. Then make the weight W double of I, and sliding I back to balance it, there mark it with 2. Do the same at 3, 4, 5, &c, by making W equal to 3, 4, 5, &c, times I; and the beam is finished. Then, to find the weight of any body b by the steelyard; take off the weight W, and hang on the body b at A; then slide the weight I backward and forward till it just balance the body b, which suppose to be at the number 5; then is b equal to 5 times the weight of I. So, if I be one pound, then b is 5 pounds; but if I be 2 pounds, then b is 10 pounds; and so on.

OF THE WHEEL AND AXLE.

62. Prop. In the wheel-and-axle; the weight and power will be in equilibrio, when the power P is to the weight W reciprocally as the radii of the circles where they act; that is, as the radius of the axle CA, where the weight hangs, to the radius of the wheel CB, where the power acts. That is, P : W :: CA : CB.

Here the cord, by which the power P acts, goes about the circumference of the wheel, while that of the weight W goes round its axle, or another smaller wheel, attached to the larger, and having the same axis or centre C. So that BA is a lever moveable about the point C, the power P acting always at the distance BC, and the weight W at the distance CA; therefore P : W :: CA : CB.

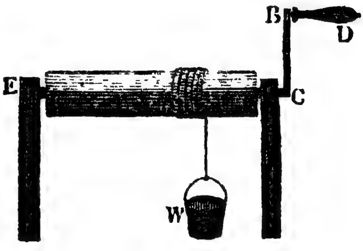


63. *Corol.* 1. If the wheel be put in motion; then, the spaces moved being as the circumferences, or as the radii, the velocity of W will be to the velocity of P, as CA to CB; that is, the weight is moved as much slower, as it is heavier than the power; so that what is gained in power, is lost in time. And this is the universal property of all machines and engines.

64. *Corol.* 2. If the power do not act at right angles to the radius C*b*, but obliquely; draw CD perpendicular to the direction of the power; then, by the nature of the lever, P : W :: OA : CD.

SCHOLIUM.

65. To this mechanical power belong all turning or wheel machines, of different radii. Thus, in the roller turning on the axis or spindle CE, by the handle CBD; the power applied at B is to the weight W on the roller, as the radius of the roller is to the radius CB of the handle.



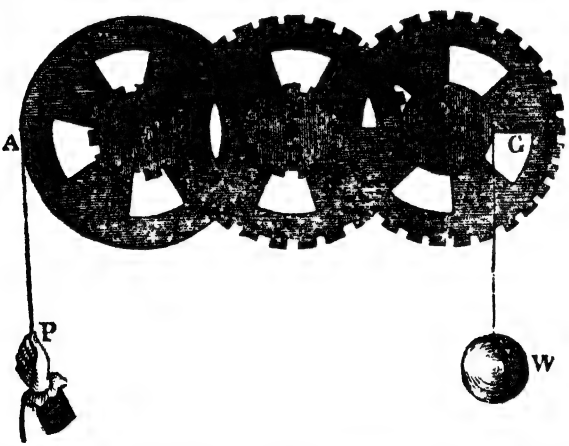
66. And the same for all cranes, capstans, windlasses,

and such like; the power being to the weight, always as the radius or lever at which the weight acts, to that at which the power acts; so that they are always in the reciprocal ratio of their velocities. And to the same principle may be referred the gimblet and augur for boring holes.

67. But all this, however, is on supposition that the ropes or cords, sustaining the weights, are of no sensible thickness. For, if the thickness be considerable, or if there be several folds of them, over one another, on the roller or barrel; then we must measure to the middle of the outermost rope, for the radius of the roller; or, to the radius of the roller, we must add half the thickness of the cord, when there is but one fold.

68. The wheel-and-axle has a great advantage over the simple lever, in point of convenience. For a weight can be raised but a little way by the lever; whereas, by the continual turning of the wheel and roller, the weight may be raised to any height, or from any depth.

69. By increasing the number of wheels too, the power may be multiplied to any extent, making always the less wheels to turn greater ones, as far as we please: and this is commonly called Tooth and Pinion Work, the teeth of one circumference working in the rounds or pinions of another, to turn the wheel. And then, in case of an equilibrium, the power is to the weight, as the continual product of the radii



of all the axles, to that of all the wheels. So, if the power P turn the wheel Q, and this turn the small wheel or axle R, and this turn the wheel S, and this turn the axle T, and this turn the wheel V; and this turn the axle X, which raises the

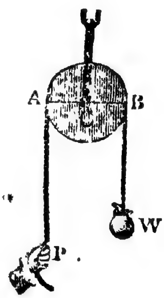
weight W; then $P : W :: CB . DE . FG : AC . BD . EF$. And in the same proportion is the velocity of W slower than that of P. Thus, if each wheel be to its axle, as 10 to 1; then $P : W :: 1^3 : 10^3$ or as 1 to 1000. So that a power of one pound will balance a weight of 1000 pounds; but then, when put in motion, the power will move 1000 times faster than the weight.

OF THE PULLEY.

70. A PULLEY is a small wheel, commonly made of wood or brass, which turns about an iron axis passing through the centre, and fixed in a block, by means of a cord passed round its circumference, which serves to draw up any weight. The pulley is either single, or combined together, to increase the power. It is also either fixed or moveable, according as it is fixed to one place, or moves up and down with the weight and power.

71. PROP. If a power sustain a weight by means of a fixed pulley: the power and weight are equal.

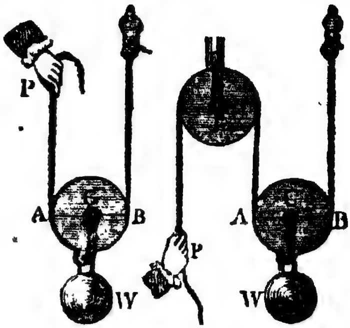
For through the centre C of the pulley draw the horizontal diameter AB: then will AB represent a lever of the first kind, its prop being the fixed centre C; from which the points A and B, where the power and weight act, being equally distant, the power P is consequently equal to the weight W.



72. *Corol.* Hence, if the pulley be put in motion, the power P will descend as fast as the weight W ascends. So that the power is not increased by the use of the fixed pulley, even though the rope go over several of them. It is, however, of great service in the raising of weights, both by changing the direction of the force, for the convenience of acting, and by enabling a person to raise a weight to any height without moving from his place, and also by permitting a great many persons at once to exert their force on the rope at P, which they could not do to the weight itself; as is evident in raising the hammer or weight of a pile-driver, as well as on many other occasions.

73. PROP. If a power sustain a weight by means of one

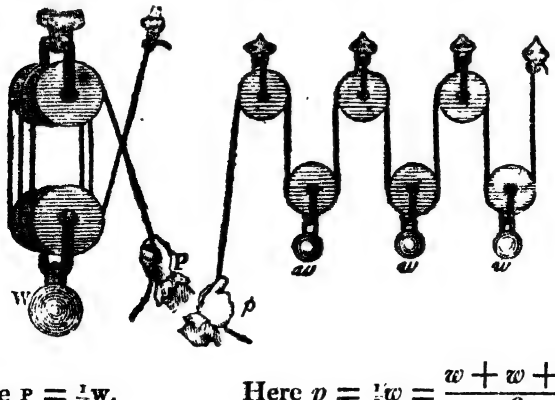
moveable pulley; the power is but half the weight, if the portions of the sustaining cord are parallel to each other.



For, here AB may be considered as a lever of the second kind, the power acting at A, the weight at C, and the prop or fixed point at B; and because P : W :: CB : AB, and CB $= \frac{1}{2}$AB, therefore P $= \frac{1}{2}$W, or W $=$ 2P.

74. *Corol.* 1. Hence it is evident, that, when the pulley is put in motion, the velocity of the power will be double the velocity of the weight, as the point P moves twice as fast as the point C and weight W rises. It is also evident, that the fixed pulley F makes no difference in the power P, but is only used to change the direction of it, from upwards to downwards.

75. *Corol.* 2. Hence we may estimate the effect of a combination of any number of fixed and moveable pulleys; by which we shall find that every cord going over a moveable pulley always adds 2 to the power; since each moveable pulley's rope bears an equal share of the weight; while each rope that is fixed to a pulley, only increases the power by unity.



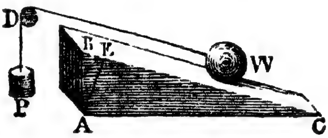
Here P $= \frac{1}{6}$W. Here $p = \frac{1}{2}w = \dfrac{w + w + w}{6}$.

Note.—If the portions of the sustaining cords between the pulleys are not parallel, the forces will be reduced upon the principle of art. 31.

OF THE INCLINED PLANE.

76. The Inclined Plane, is a plane inclined to the horizon, or making an angle with it. It is often reckoned one of the simple mechanic powers; and the double inclined plane makes the wedge. It is employed to advantage in raising heavy bodies in certain situations, diminishing their weights by laying them on the inclined planes.

77. Prop. The power gained by the inclined plane, is in proportion as the length of the plane is to its height. That is, when a weight W is sustained on an inclined plane BC, by a power P acting in the direction DW, parallel to the plane; then the weight W, is in proportion to the power P, as the length of the plane is to its height; that is, W : P :: BC : AB.



For, draw AE perp. to the plane BC, or to DW. Then we are to consider that the body W is sustained by three forces, viz. 1st, its own weight or the force of gravity, acting perp. to AC, or parallel to BA; 2d, by the power P, acting in the direction WD, parallel to BC, or BE; and 3dly, by the re-action of the plane, perp. to its face, or parallel to the line EA. But when a body is kept in equilibrio by the action of three forces, it has been proved, (art. 30.) that the intensities of these forces are proportional to the sides of the triangle ABE, made by lines drawn in the directions of their actions; therefore those forces are to one another as the three lines - - AB, BE, AE; that is,
the weight of the body W is as the line AB,
the power P is as the line - - BE,
and the pressure on the plane as the line AE.
But the two triangles ABE, ABC are equiangular, and have therefore their like sides proportional; that is,
the three lines - - - - - AB, BE, AE,
are to each other respectively as the three BC, AB, AC,
or also as the three - - - AC, AE, CE,
which therefore are as the three forces W, P, p,
where p denotes the pressure on the plane. That is, W : P :: BC : AB, or the weight is to the power, as the length of the plane is to its height.

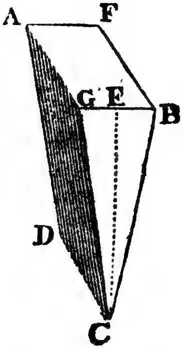
See more on the Inclined Plane in the Dynamics.

78. *Scholium.* The Inclined Plane comes into use in some situations in which the other mechanical powers cannot be conveniently applied, or in combination with them. As, in sliding heavy weights either up or down a plank or other plane laid sloping: or letting large casks down into a cellar, or drawing them out of it. Also, in removing earth from a lower situation to a higher by means of wheelbarrows, or otherwise, as in making fortifications, &c.; inclined planes, made of boards are employed. Rail-roads, on inclined planes, serve often to convey coals from the mouth of a mine.

Of all the various directions of drawing bodies up an inclined plane, or sustaining them on it, the most favourable is where it is parallel to the plane BC, and passing through the centre of the weight; a direction which is easily given to it, by fixing a pulley at D, so that a cord passing over it, and fixed to the weight, may act or draw parallel to the plane. In every other position, it would require a greater power to support the body on the plane, or to draw it up. For if one end of the line be fixed at W, and the other end inclined down towards B, below the direction WD, the body would be drawn down against the plane, and the power must be increased in proportion to the greater difficulty of the traction. And, on the other hand, if the line were carried above the direction of the plane, the power must be also increased; but here only in proportion as it endeavours to lift the body off the plane.

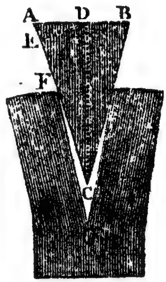
If the length BC of the plane be equal to any number of times its perp. height AB, as suppose 3 times; then a power P of 1 pound, hanging freely, will balance a weight W of 3 pounds, laid on the plane; and a power P of 2 pounds, will balance a weight W of 6 pounds; and so on, always 3 times as much. But then if they be set moving, the perp. descent of the power P, will be equal to 3 times as much as the perp. ascent of the weight W. For, though the weight W ascends up the direction of the oblique plane, BC, just as fast as the power P descends perpendicularly, yet the weight rises only the perp. height AB, while it ascends up the whole length of the plane BC, which is three times as much; that is, for every foot of the perp. rise of the weight, it ascends 3 feet up in the direction of the plane, and the power P descends just as much, or 3 feet.

OF THE WEDGE.



79. THE WEDGE is a piece of wood or metal, in form of half a rectangular prism. AF or BG is the breadth of its back; CE its height; GC, BC its sides; and its end GBC is composed of two equal inclined planes GCE, BCE.

80. PROP. When a wedge is in equilibrio; the power acting against the back, is to the force acting perpendicularly against either side, as the breadth of the back AB is to the length of the side AC or BC.



For, any three forces, which sustain one another in equilibrio, are as the corresponding sides of a triangle drawn perpendicular to the directions in which they act. But AB is perp. to the force acting on the back, to urge the wedge forward; and the sides AC, BC are perp. to the forces acting on them; therefore the three forces are as AB, AC, BC.

81. *Corol.* The force on the back AB,
Its effect in direct. perp. to AC, AC,
And its effect parallel to AB; DC,
are as the three lines which are per. to them.

And therefore the thinner a wedge is, the greater is its effect, in splitting any body, or in overcoming any resistance against the sides of the wedge.

SCHOLIUM.

82. But it must be observed, that the resistance, or the forces above-mentioned, respect one side of the wedge only. For if those against both sides be taken in, then, in the foregoing proportions, we must take only half the back AD, or else we must take double the line AC or DC. Various other theories of the wedge are given by different authors, but they need not here be detailed, on account of the irregularities introduced by friction.

In the wedge, the friction against the sides is very great, at least equal to the force to be overcome, because the wedge retains any position to which it is driven; and therefore the resistance is doubled by the friction. But then the wedge has a great advantage over all the other powers, arising from the force of percussion or blow with which the back is struck, which is a force incomparably greater than any dead weight or pressure, such as is employed in other machines. And accordingly we find it produces effects vastly superior to those of any other power; such as the splitting and raising the largest and hardest rocks, the raising and lifting the largest ship, by driving a wedge below it, which a man can do by the blow of a mallet: and thus it appears that the small blow of a hammer, on the back of a wedge, is incomparably greater than any mere pressure, and will overcome it.

OF THE SCREW.

83. The Screw is one of the six mechanical powers, chiefly used in pressing or squeezing bodies close, though sometimes also in raising weights.

The screw is a spiral thread or groove cut round a cylinder, and every where making the same angle with the length of it. So that if the surface of the cylinder, with this spiral thread on it, were unfolded and stretched into a plane, the spiral thread would form a straight inclined plane, whose length would be to its height, as the circumference of the cylinder, is to the distance between two threads of the screw: as is evident by considering that, in making one round, the spiral rises along the cylinder the distance between the two threads.

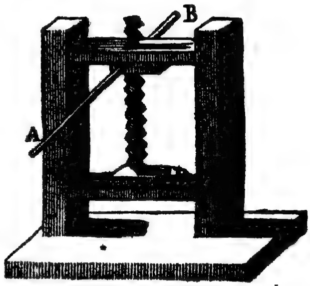
84. Prop. The energy of a power applied to turn a screw round, is to the force with which it presses upward or downward, setting aside the friction, as the distance between two threads, is to the circumference where the power is applied.

The screw being an inclined plane, or half wedge, whose height is the distance between two threads, and its base the circumference of the screw; and the force in the horizontal direction, being to that in the vertical one, as the lines perpendicular to them, namely, as the height of the plane, or distance of the two threads, is to the base of the plane, or circumference of the screw; therefore the power is to the pressure, as the distance of two threads is to that circumference. But, by means of a handle or lever, the gain in

power is increased in the proportion of the radius of the screw to the radius of the power, or length of the handle, or as their circumferences. Therefore, finally, the power is to the pressure, as the distance of the threads, is to the circumference described by the power.

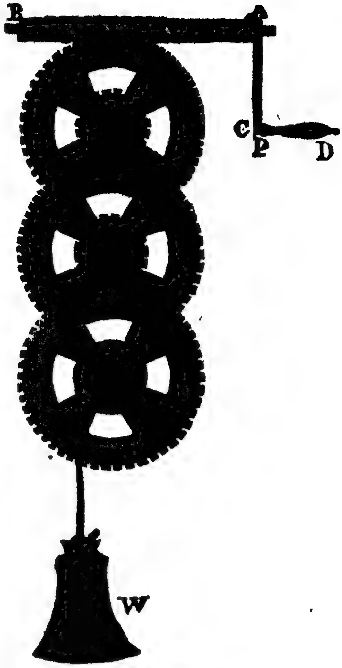
85. *Corol.* When the screw is put in motion; then the power is to the weight which would keep it in equilibrio, as the velocity of the latter is to that of the former; and hence their two momenta are equal, which are produced by multiplying each weight or power by its own velocity. So that this is a general property in all the mechanical powers, namely, that the momentum of a power is equal to that of the weight which would balance it in equilibrio; or that each of them is reciprocally proportional to its velocity.

SCHOLIUM.



86. Hence we can easily compute the force of any machine turned by a screw. Let the annexed figure represent a press driven by a screw, whose threads are each a quarter of an inch asunder; and let the screw be turned by a handle of 4 feet long, from A to B; then, if the natural force of a man, by which he can lift, pull, or draw, be 150 pounds; and it be required to determine with what force the screw will press on the board at D, when the man turns the handle at A and B, with his whole force. Then the diameter AB of the power being 4 feet, or 48 inches, its circumference is $48 \times 3{\cdot}1416$ or $150\frac{4}{5}$ nearly; and the distance of the threads being $\frac{1}{4}$ of an inch; therefore the power is to the pressure, as 1 to $603\frac{1}{5}$; but the power is equal to 150lb; theref. as $1 : 603\frac{1}{5} :: 150 : 90480$; and consequently the pressure at D is equal to a weight of 90480 pounds, independent of friction.

87. Again, if the endless screw AB be turned by a handle AC of 20 inches, the threads of the screw being distant half an inch each; and the screw turns a toothed wheel E, whose pinion L turns another wheel F, and the pinion M of this another wheel G, to the pinion or barrel of which is hung a weight W; it is required to determine what weight the man will be able to raise, working at the handle C; supposing



the diameters of the wheels to be 18 inches, and those of the pinions and barrel 2 inches; the teeth and pinions being all of a size.

Here $20 \times 3{\cdot}1416 \times 2 = 125{\cdot}664$, is the circumference of the power.

And $125{\cdot}664$ to $\frac{1}{2}$, or $251{\cdot}328$ to 1, is the force of the screw alone.

Also, 18 to 2, or 9 to 1, being the proportion of the wheels to the pinions; and as there are three of them, therefore 9^3 to 1^3, or 729 to 1, is the power gained by the wheels.

Consequently $251{\cdot}328 \times 729$ to 1, or $183218\frac{1}{9}$ to 1 nearly, is the ratio of the power to the weight, arising from the advantage both of the screw and the wheels.

But the power is 150lb; therefore $150 \times 183218\frac{1}{9}$, or 27482716 pounds, is the weight the man can sustain, which is equal to 12269 tons weight.

But the power has to overcome, not only the weight, but also the friction of the screw, which is very great, in some cases equal to the weight itself, since it is sometimes sufficient to sustain the weight, when the power is taken off.

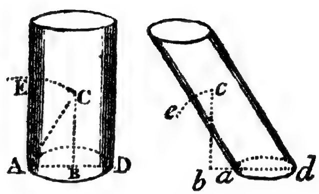
88. Upon the same principle the advantage of any other combination of the mechanical powers may be computed: allowance, however, being always to be made for stiffness of cords, friction, and other causes of resistance.

ON THE CENTRE OF GRAVITY.

89. **The Centre of Gravity** of a body, or of a system of bodies, is a certain point within it, or connected with it, on which the body being freely suspended, it will rest in any

position, and that centre will always tend to descend to the lowest place to which it can get, when it is not the point of suspension.

90. Prop. If a perpendicular to the horizon, from the centre of gravity of any body, fall within the base of the body, it will rest in that position; but if the perpendicular fall out of the base, the body will not rest in that position, but will fall down.



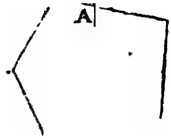
For, if CB, be the perp. from the centre of gravity C, within the base: then the body cannot fall over towards A; because, in turning on the point A, the centre of gravity C would describe an arc which would rise from C to E; contrary to the nature of that centre, which only rests permanently when in the lowest place. For the same reason, the body will not fall towards D. And therefore it will stand in that position.

But if the perpendicular fall out of the base, as *cb*, then the body will fall over on that side: because, in turning on the point *a*, the centre *c* descends by describing the descending arc *ce*.

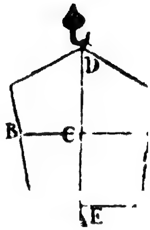
91. *Corol.* 1. If a perpendicular, drawn from the centre of gravity, fall just on the extremity of the base, the body may stand; but any the least force will cause it to fall that way. And the nearer the perpendicular is to any side, or the narrower the base is, the easier it will be made to fall, or be pushed over that way; because the centre of gravity has the less height to rise: which is the reason that a globe is made to roll on a smooth plane by any the least force. But the nearer the perpendicular is to the middle of the base, or the broader the base is, the firmer the body stands.

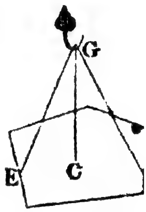
92. *Corol.* 2. Hence if the centre of gravity of a body be supported, the whole body is supported. And the place of the centre of gravity may, in many inquiries, be accounted the place of the body; for into that point the whole matter of the body may be supposed to be collected, and therefore all the force also with which it endeavours to descend.

93. *Corol.* 3. From the property which the centre of gravity has, of tending to descend to the lowest point, is derived an easy mechanical method of finding that centre.



Thus, if the body be hung up by any point A, and a plumb line AB be hung by the same point, it will pass through the centre of gravity; because that centre is not in the lowest point till it fall in the plumb line. Mark the line AB on it. Then hang the body up by any other point D, with a plumb line DE, which will also pass through the centre of gravity, for the same reason as before; and therefore that centre must be at C where the two plumb lines cross each other.





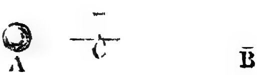
94. Or, if the body be suspended by two or more cords GF, GH, &c. then a plumb line from the point G, will cut the body in the centre of gravity C.

95. Likewise, because a body rests when its centre of gravity is supported, but not else; we hence derive another easy method of finding that centre mechanically. For, if the body be laid on the edge of a prism, or over one side of a table, and moved backward and forward till it rest, or balance itself; then is the centre of gravity just over the line of the edge. And if the body be then shifted into another position, and balanced on the edge again, this line will also pass by the centre of gravity; and consequently the intersection of the two will indicate the place of the centre itself.

The place of the centre of gravity may be investigated, from its analogy to the centre of parallel forces; but the following method is adopted here, as in some respects easier of comprehension.

96. Prop. The common centre of gravity C of any two bodies A, B, divides the line joining their respective centres, into two parts, which are reciprocally as the bodies.

That is, AC : BC :: B : A.



For, if the centre of gravity C be supported, the two bodies A and B will be supported, and will

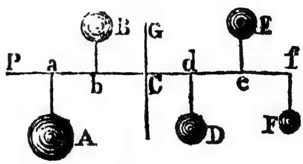
rest in equilibrio. But, by the nature of the lever, when two bodies are in equilibrio about a fixed point C, they are reciprocally as their distances from that point; therefore A : B :: CB : CA.

97. *Corol.* 1. Hence AB : AC :: A + B : B; or, the whole distance between the two bodies, is to the distance of either of them from the common centre, as the sum of the bodies is to the other body.

98. *Corol.* 2. Hence also, CA . A = CB . B; or the two products are equal, which are made by multiplying each body into its distance from the centre of gravity.

99. *Corol.* 3. As the centre C is pressed with a force equal to both the weights A and B, while the points A and B are each pressed with the respective weights A and B; therefore, if the two bodies be both united in their common centre C, and only the ends A and B of the line AB be supported, each will still bear, or be pressed by the same weights A and B as before. So that, if a weight of 100lb. be laid on a bar at C, supported by two men at A and B, distant from C, the one 4 feet, and the other 6 feet; then the nearer will bear the weight of 60lb, and the farther only 40lb. weight. This should be noted as a principle of extensive application.

100. *Corol.* 4. Since the effect of any body to turn a lever about the fixed point C, is as that body and as its distance from that point; therefore, if C be the common centre of gravity of all the bodies A, B, D, E, F, placed in the straight line AF; then is CA . A + CB . B = CD . D + CE . E + CF . F; or, the sum of the products on one side, equal to the sum of the products on the other, made by multiplying each body into its distance from that centre. And if several bodies be in equilibrio on any straight lever, then the prop is in the centre of gravity.

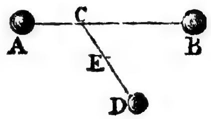


101. *Corol.* 5. And though the bodies be not situated in a straight line, but scattered about in any promiscuous manner, the same property as in the last corollary still holds true, if perpendiculars to any line whatever *af* be drawn through the several bodies, and their common centre of gravity, namely, that *ca* . A + *cb* . B = *cd* . D + *ce* . E + *cf* . F. For

the bodies have the same effect on the line *af*, to turn it about the point C, whether they are placed at the points *a*, *b*, *d*, *e*, *f*; or in any part of the perpendiculars A*a*, B*b*, D*d*, E*e*, F*f*.

102. PROP. If there be three or more bodies, and if a line be drawn from any one body D to the centre of gravity of the rest C; then the common centre of gravity E of all the bodies, divides the line CD into two parts in E, which are reciprocally proportional as the body D to the sum of all the other bodies.

That is, CE : ED : : D : A + B &c.

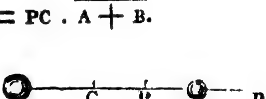


For, suppose the bodies A and B to be collected into the common centre of gravity C, and let their sum be called S. Then, by the last prop. CE : ED : : D : S or A + B &c.

Corol. Hence we have a method of finding the common centre of gravity of any number of bodies; namely, by first finding the centre of any two of them, then the centre of that centre and a third, and so on for a fourth, or fifth, &c.

103. PROP. If there be taken any point P, in the line passing through the centres of two bodies; then the sum of the two products, of each body multiplied into its distance from that point, is equal to the product of the sum of the bodies multiplied into the distance of their common centre of gravity C from the same point P.

That is, $PA \cdot A \pm PB \cdot B = PC \cdot \overline{A + B}$.



For, by art. 98th, $CA \cdot A = CB \cdot B$, that is, $(PA - PC) \cdot A = (PC \pm PB) \cdot B$; therefore, by adding, $PA \cdot A \pm PB \cdot B = PC \cdot (A + B)$.

104. *Corol.* 1. Hence, the two bodies A and B have the same force to turn the lever about the point P, as if they were both placed in C their common centre of gravity.

Or, if the line, with the bodies, move about the point P; the sum of the momenta of A and B, is equal to the momentum of the sum S or A + B placed at the centre C.

105. *Corol.* 2. The same is also true of any number of bodies whatever, as will appear by cor. 4, art. 100. namely, $PA \cdot A + PB \cdot B + PD \cdot D$ &c. $= PC \cdot (A + B + D$ &c.$)$ where P is in any point whatever of the line AC.

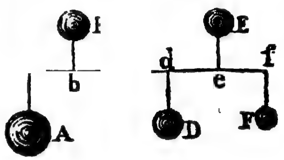
And, by cor. 5, art. 101, the same thing is true when the

bodies are not placed in that line, but any where in the perpendiculars passing through the points A, B, D, &c; namely, $Pa \cdot A + Pb \cdot B + Pd \cdot D$ &c. $= PC \cdot (A + B + D$ &c.)

106. *Corol.* 3. And if a plane pass through the point P perpendicular to the line CP; then the distance of the common centre of gravity from that plane, is

$$PC = \frac{Pa \cdot A + Pb \cdot B + Pd \cdot D \text{ \&c.}}{A + B + D \text{ \&c.}},$$ that is, equal to the sum of all the moments divided by the sum of all the bodies. Or, if A, B, D, &c, be the several particles of one mass or compound body; then the distance of the centre of gravity of the body, below any given point P, is equal to the forces of all the particles divided by the whole mass or body, that is, equal to all the $Pa \cdot A$, $Pb \cdot B$, $Pd \cdot D$, &c, divided by the body or sum of particles A, B, D, &c.

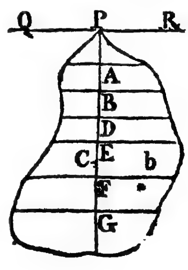
107. PROP. To find the centre of gravity of any body, or of any system of bodies.



Through any point P draw a plane, and let Pa, Pb, Pd, &c, be the distance of the bodies A, B, D, &c, from the plane; then, by the last cor. the distance of the common centre of gravity from the plane, will be

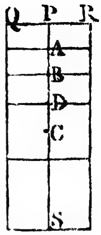
$$PC = \frac{Pa \cdot A + Pb \cdot B + Pd \cdot D \text{ \&c}}{A + B + D \text{ \&c}}.$$

108. Or, if b be any body, and QPR any plane; draw PAB &c, perpendicular to QR, and through A, B, &c, draw innumerable sections of the body b parallel to the plane QR. Let s denote any one of these sections, and $d = PA$, or PB, &c, its distance from the plane QR. Then will the distance of the centre of gravity of the body from the plane be



$$PC = \frac{\text{sum of all the } ds}{b}.$$ And if the distance be thus found for two intersecting planes, they will give the point in which the centre is placed.

109. But the distance from one plane is sufficient for any regular body, because it is evident that, in such a figure, the centre of gravity is in the axis, or line passing through the centres of all the parallel sections.



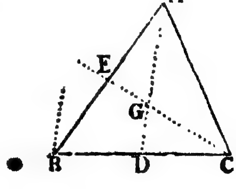
Thus, if the figure be a parallelogram, or a cylinder, or any prism whatever; then the axis or line, or plane PS, which bisects all the sections parallel to QR, will pass through the centre of gravity of all those sections, and consequently through that of the whole figure C. Then, all the sections s being equal, and the body $b = \text{PS} \cdot s$, the distance of the centre will be PC =

$$\frac{\text{PA} \cdot s + \text{PB} \cdot s + \&\text{c}}{b} = \frac{\text{PA} + \text{PB} + \text{PD}\,\&\text{c}}{\text{PS} \cdot s} \times s = \frac{\text{PA} + \text{PB} + \&\text{c}}{\text{PS}}.$$

But PA + PB + &c, is the sum of an arithmetical progression, beginning at 0, and increasing to the greatest term PS, the number of the terms being also equal to PS; therefore the sum $\text{PA} + \text{PB} + \&\text{c} = \frac{1}{2}\text{PS} \cdot \text{PS}$; and consequently $\text{PC} = \frac{\frac{1}{2}\text{PS} \cdot \text{PS}}{\text{PS}} = \frac{1}{2}\text{PS}$; that is, the centre of gravity is in the middle of the axis of any figure whose parallel sections are equal.

110. In other figures, whose parallel sections are not equal, but varying according to some general law, it will not be easy to find the sum of all the $\text{PA} \cdot s$, $\text{PB} \cdot s'$, $\text{PD} \cdot s''$, &c, except by the general method of Fluxions; which case therefore will be best reserved, till we come to treat of that doctrine. It will be proper, however, to add here some examples of another method of finding the centre of gravity of a triangle, or any other right-lined plane figure.

111. Prop. To find the centre of gravity of a triangle.



From any two of the angles draw lines AD, CE, to bisect the opposite sides; so will their intersection G be the centre of gravity of the triangle.

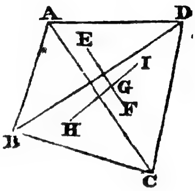
For, because AD bisects BC, it bisects also all its parallels, namely, all the parallel sections of the figure: therefore AD passes through the centres of gravity of all the parallel sections or component parts of the figure; and consequently the centre of gravity of the whole figure lies in the line AD. For the same reason, it also lies in the line CE. Consequently it is in their common point of intersection G.

112. *Corol.* The distance of the point G, is $\text{AG} = \frac{2}{3}\text{AD}$, and $\text{CG} = \frac{2}{3}\text{CF}$: or $\text{AG} = 2\text{GD}$, and $\text{CG} = 2\text{GE}$.

For, draw BF parallel to AD, and produce CE to meet it

in F. Then the triangles AEG, BEF are similar, and also equal, because AE = BE; consequently AG = BF. But the triangles CDG, CBF are also equiangular, and CB being = 2CD, therefore BF = 2GD. But BF is also = AG; consequently AG = 2GD or $\frac{2}{3}$AD. In like manner, CG = 2GE or $\frac{2}{3}$CE.

113. PROP. To find the centre of gravity of a trapezium.



Divide the trapezium ABCD into two triangles, by the diagonal BD, and find E, F, the centres of gravity of these two triangles; then shall the centre of gravity of the trapezium lie in the line EF connecting them. And therefore if EF be divided, in G, in the alternate ratio of the two triangles, namely, EG : GF : : triangle BCD : triangle ABD, then G will be the centre of gravity of the trapezium.

114. Or, having found the two points E, F, if the trapezium be divided into two other triangles BAC, DAC, by the other diagonal AC, and the centres of gravity H and I of these two triangles be likewise found; then the centre of gravity of the trapezium will also lie in the line HI.

So that, lying in both the lines, EF, HI, it must necessarily lie in their intersection G.

115. And thus we are to proceed for a figure of any greater number of sides, finding the centres of their component triangles and trapeziums, and then finding the common centre of every two of these, till they be all reduced into one only.

PROBLEMS FOR EXERCISE.

1. Find, geometrically, the centre of gravity of a trapezoid.

2. Find, geometrically, the centre of gravity of a triangular pyramid.

3. Infer, thence, the centre of gravity of any pyramid.

4. Find, algebraically, the centre of gravity of the frustum of a pyramid.

5. Let a sphere whose diameter is 4 inches, and a cone whose altitude is 8 inches, and diameter of its base 8 inches, be fastened upon a thin wire which shall pass through the centre of the globe and the axis of the cone; let the vertex of the cone be toward the sphere, and let its distance from

the sphere's surface be 12 inches. Required the place of their common centre of gravity.

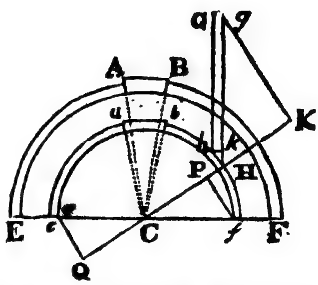
6. Demonstrate 1st, That the surface produced by a plane line or curve by revolving about an axis in the plane of that curve, is equal to the product of the generating line or curve into the path described by the centre of gravity.

And 2dly, That the solid produced by the revolution of a plane figure about an axis posited in the plane of that figure, is equal to the product of the generating surface into the circumference described by the centre of gravity.

ON THE EQUILIBRIUM OF ARCHES.

116. A very interesting department of the science of Statics, is that which relates to the stability of arches, as introduced in the construction of bridges, powder-magazines, &c. Every such structure is a system of forces, and the examination of its firmness, therefore, requires the application of the general principles of equilibrium. We shall here present a few useful propositions in elucidation of the more received theories.

117. PROP. The force of a voussoir depending on the magnitude of the angle formed by its sides, the impelling force, and the resistance to be overcome, is on the first account directly as the radius of curvature of the arch at that point, on the second as the square of the sine of the angle included between the tangent of the curve at the given point and the vertical passing through that point, and on the third, as the sine of the same angle.



1. Let EABF, *eabf*, be two similar concentric curves, and AB, *ab*, two voussoirs similarly situated; whose sides perpendicular to the curve converge to the centre C. The forces of these voussoirs considered as portions of wedges, are inversely as the sines of the half vertical angles (schol. art. 82.) or, because each wedge occupies an equal portion of its respective arch, directly as the radii of curvature.

2dly, Let Hh be the invariable breadth of the voussoirs on the arch $eabf$, GgHh the incumbent weight, which, since GH is supposed given, is as the breadth hk, or as the sine of the angle hHk: by the resolution of the force gH into two hH, HK, the latter is the force impelling the voussoir to split the arch, which, since gH is given, varies as the sine of HgK, or hHk: wherefore, the force impelling the voussoir is as the square of the sine of hHk.

3dly, The wedge impelled in a direction perpendicular to the curve endeavours to split the arch, and therefore to move one segment about the fulcrum e, the other about the fulcrum f. Hence the force of the voussoir acting on the levers Hf, He, being as either of the perpendiculars fP, eQ, is as the sine of the angle fCP or hHk.

We have supposed the centre of curvature of the arches at the points A, a, h, H, to be at C: but this is merely to prevent the figure from being too complex, and makes no alteration in the nature of the demonstration.

Corol. Hence, if the height of the wall incumbent on any point H of the intrados is inversely as the cube of the sine of hHk into radius of curvature at that point, or directly as cube of the secant of the angle formed by hH and the horizon, and inversely as the radius of curvature, all the voussoirs will endeavour to split the arch with equal forces, and will be in perfect equilibrium with each other.

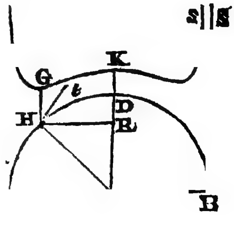
The general expression, therefore, for the thickness GH over any point of an arch, is

$$\text{GH} = \text{sec.}^3 \text{ elev}^{\text{n}}. \text{ at H} \times \frac{ar}{\text{R}}$$

where r = rad. of curvature at the vertex
a = thickness of material there
R = rad. of curvature at H.

The radii of curvature for the different curves are determinable by the method of fluxions, or by other means: they are here supposed known.

1. Suppose, for example, it were required to find the requisite thickness over any point of a circular arc, to ensure equilibration, the thickness a = DK, at the crown of the arch being given.



Here, rad. of curv. at H
= rad. of curv. at D
that is R = r.
and sec. RHT = sec. arc. DH.

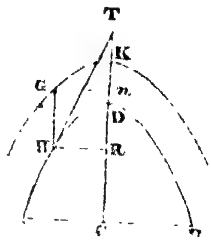
Conseq. GH $=$ sec.^{3} DH $\times \frac{ar}{R} =$ sec.^{3} DH . a.

Hence we have a convenient logarithmic expression for computation; viz.

Log. DK $+$ 3 log. sec. DH $=$ log. GH.

In this example, the curve of equilibration, GKS, runs up to an infinite height over B, the springing of a semicircular arch. But over a portion of 30^0 or 35^0 on each side the vertex, as DH, the curve KG of the extrados accords very well with what would be required for a roadway.

Ex. 2. Determine the requisite thickness for equilibration over any point of a parabola.



Here if DR$=x$, RH $=y$, R$=\frac{(4x+p)^{\frac{3}{2}}}{2\sqrt{p}}$; which, at the vertex, where x vanishes becomes $r = \frac{1}{2}p$

RT $= 2x$, RH $= y = \sqrt{px}$, TH $= \sqrt{\text{HR}^2 + \text{RT}^2} = \sqrt{px + 4x^2}$

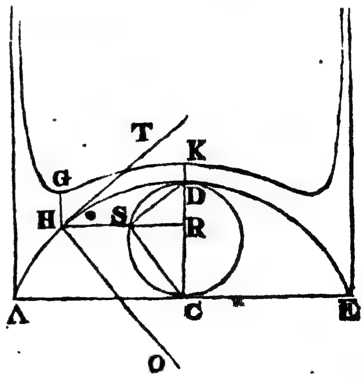
sec. THR $= \frac{\text{TH}}{\text{HR}} = \sqrt{\frac{px + 4x^2}{px}} = \sqrt{\frac{p+4x}{p}}$

$\therefore$ GH $=$ sec.^{3} THR . $\frac{r}{R}\, a$

$$= \frac{(p+4x)^{\frac{3}{2}}}{p^{\frac{3}{2}}} \cdot \tfrac{1}{2}p \cdot \frac{2\sqrt{p}}{(p+4x)^{\frac{3}{2}}}\, a = a = \text{KD}.$$

So that the extrados is a parabola equal to the intrados, and every where vertically equidistant from it.

Ex. 3. To determine the requisite thickness over any point of a cycloidal arch.



Here, putting DK $= a$, DR$=x$, DC$=d$; we have, from the known properties of the cycloid, the tangent HT parallel to the corresponding chord SD, or angle THR $=$ $\angle$ DSR, SD $= \sqrt{dx}$, SR $= \sqrt{dx - x^2}$;

R (parallel to SC) $= \text{HO} = 2\text{SC} = 2\sqrt{d^2 - dx}$; $r = 2\text{CD} = 2d$:

$$\text{sec. THR} = \frac{\text{SD}}{\text{SR}} = \sqrt{\frac{dx}{dx - x^2}} = \sqrt{\frac{d}{d-x}}.$$

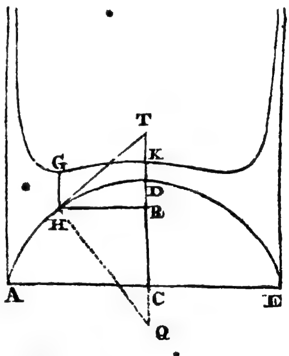
$$\text{GH} = \text{sec.}^3\ \text{THR} \cdot \frac{r}{\text{R}} \cdot a$$

$$= \frac{d^{\frac{3}{2}}}{(d-x)^{\frac{3}{2}}} \cdot 2d \cdot \frac{a}{2(d^2 - dx)^{\frac{1}{2}}},$$

$$= \frac{d^{\frac{3}{2}}}{(d-x)^{\frac{3}{2}}} \cdot \frac{2da}{2d^{\frac{1}{2}}(d-x)^{\frac{1}{2}}},$$

$$= \frac{d^{\frac{4}{2}}}{(d-x)^{\frac{4}{2}}}\, a = \frac{ad^2}{(d-x)^2} = \frac{\text{DK} \cdot \text{CD}^2}{\text{CR}^2}.$$

By computing the value of GH for several corresponding values of DR, and CR, and thence constructing the extrados by points, it will, as in the figure, appear analogous to that for the circle, but rather flatter till it approach the extremities of the arch, where the curve runs off to infinity, as in the case for the circle.



EXAM. 4. To determine the requisite thickness over any point of an elliptical arch.

Here, taking x, y, and a, as before, take $\text{AC} = t$, $\text{DC} = c$, $\text{HQ} = \pi$, being perpendicular to the tangent HT. Then, by the property of the ellipse,

$$\text{DC}^2 : \text{AC}^2 :: \text{CR} : \text{QR},$$

$$\text{or, } c^2 : t^2 :: c - x : \frac{t^2}{c^2}(c - x) = \text{QR}.$$

Also, sec. THR $=$ sec. HQR $= \dfrac{\text{HQ}}{\text{QR}} = \pi \div \dfrac{t^2}{c^2}(c-x) = \dfrac{\pi c^2}{t^2(c-x)}$.

Radius of curvature at H $=$ R $= \dfrac{4\pi^3}{p^2}$, p being the parameter to CD $= \dfrac{2t^2}{c}$:

$$\therefore \text{R} = \frac{4\pi^3 c^2}{4t^4} = \frac{\pi^3 c^2}{t^4}; \text{ and } r \text{ (rad. curv. at D)} = \frac{t^2}{c}.$$

Whence, lastly, $\text{GH} = \sec.^3 \text{THR} \cdot \frac{r}{\text{R}} \cdot a$,

$$\frac{\pi^3 c^6}{t^6 (c-x)^3} \cdot \frac{t^2}{c} \cdot \frac{t^4}{\pi^3 c^2}$$

$$= \frac{\quad}{(c-x)^3} a = \frac{\quad}{\text{CR}^3}.$$

as before, a convenient expression for logarithmic operation.

Here, again, computing values of GH for several assumed values of CR, the curve of the extrados may thence be constructed, and, like that for the cycloid, it will be found rather flatter than that for the circle, but still analogous to it.

Exam. 5. For the Catenary. (See the fig. to Exam. 2.)

Here, put $\text{DR} = x$, $\text{GR} = y$, $\text{DG} = z$, t = tension at the vertex D when the chain hangs from A and B. Then, by the nature of the curve $z^2 = 2tx + x^2$, subtang. $\text{TR} = \frac{zy}{t}$

Rad. curv. at $\text{G} = \frac{t^2 + z^2}{t} = \text{R}$, and therefore at D where z vanishes $r = t$.*

$$\text{HT} = \sqrt{\text{HR}^2 + \text{RT}^2} = \sqrt{y^2 + \frac{z^2 y^2}{t^2}},$$

$$\sec. \text{THR} = \frac{\text{TH}}{\text{HR}} = \sqrt{y^2 + \frac{z^2 y^2}{t^2}} \div y = \sqrt{1 + \frac{z^2}{t^2}} = \sqrt{\frac{t^2 + z^2}{t^2}}.$$

$$\therefore \text{GH} = \sec.^3 \text{THR} \cdot \frac{r}{\text{R}} a = \frac{(t^2 + z^2)^{\frac{3}{2}}}{t^3} \cdot t \cdot \frac{t}{t^2 + z^2} \cdot a$$

$$= \frac{(t^2 + z^2)^{\frac{1}{2}} t^2 a}{t^3} = \frac{(t^2 + z^2)^{\frac{1}{2}} a}{t} \text{ [sub.}^g \text{ for } z^2 \text{ its value.]}$$

$$= \frac{(t^2 + 2tx + x^2)^{\frac{1}{2}} a}{t} = \frac{a(t + x)}{t} = a + \frac{ax}{t}.$$

Corol. If $a = t$, or the thickness at the crown equal to a line whose weight expresses the tension,

then $\text{GH} = a + x = \text{KD} + \text{DR}$.

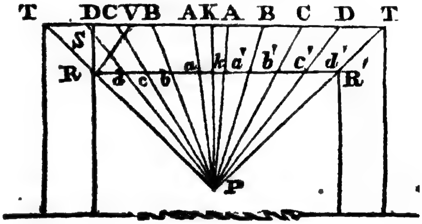
Corol. 2. If $a \gtrless t$, the exterior curve will proceed $\left\{\begin{matrix}\text{upwards} \\ \text{downwards}\end{matrix}\right\}$ both ways from K.

Corol. 3. If DK, the thickness at the crown, be very small compared with t, then will the thickness over H be nearly the same throughout: thus, suppose $a = \frac{1}{10000} t$, then $\text{GH} = a + \frac{tx}{10000t} = a + \frac{x}{10000} = a$ very nearly. Consequently, a heavy flexible cord or chain, left to adjust itself into a hanging catenary, and inverted, would support itself upon props perpendicular to the tangents at A and B.

Or $\text{K}n = a + x - (a + \frac{ax}{t}) = x - \frac{ax}{t} = (t-a)\frac{x}{t}$;

which when a vanishes becomes $= x$, or $n\text{R} = \text{GH} = a$.

118. PROP. To point out the construction, and investigate the chief properties of the plat-band, or "flat arch," as it is sometimes called.



Let RR′ be the proposed width, and Kk the proposed thickness of a plat-band. Assume a point P in the inferior prolongation of Kk the middle of the structure; and, supposing aa', ab, bc, cd, &c. the proposed thicknesses at bottom, of the truncated wedges of which the plat-band is to be constituted, let straight lines Pa'A′, PaA, PbB, PcC, &c. be drawn, they will respectively show the directions in which the mutually abutting faces of the several wedges are to be cut, so that the whole shall be an equilibrated structure.

Now, 1st, If $ak = a'k$, be taken to represent half the *weight* of the central wedge, then Pk perpendicular to it will represent the *horizontal thrust* throughout the plat-band, and consequently, the *thrust, shoot,* or *drift,* acting at R or R′.

2dly, Therefore, by assuming P nearer or farther from RR′, the thrust may be diminished or increased at pleasure.

3dly, No one of the wedges has a greater tendency to fall downwards than another; for those tendencies are throughout as their weights, each being represented by the successive lines ab, bc, cd, &c. on both sides the key-stone. The former are as the differences of the tangents ka, kb, kc, &c. to the radius Pk; and the latter are as the areas of the trape-

zoids abBA, bcCB, &c. which are as $ab+$ AB to $bc+$ BC, or as ab, to bc; the common height of all the trapezoids being equal to kK.

4thly, The *pressure* on each joint of the plat-band is *proportional to the surface of that joint.* For, pressure on aA to pressure on bB, as Pa to Pb, that is, as aA to bB; and so throughout. The pressures being exerted perpendicularly to the respective surfaces, are evidently measured by lines in the directions of those surfaces (art. 32.) when we have assumed a horizontal line for the measure of gravity, and a vertical line to measure the horizontal thrust.

5thly, Hence also it follows that in this construction the pressure upon each square inch of joint, is a constant quantity throughout; being the same upon every square inch of the face in direction aA, as upon every square inch of face in direction bB, in direction cC, &c. to the extreme abutments RT, R′T′.

These properties will not be found co-existent in any other equilibrated structure.

119. *Scholium.* Yet this construction has a limitation which it is highly important to observe. To ensure stability, the distance of the centre of gravity of the semi-vault from the vertical PK, must exceed KV, the distance from the same vertical to the intersection of RV (a perpendicular to the abutment TR) with the top TT′ of the plat-band. Unless this condition be fulfilled perpendiculars cannot be let fall from the centre of gravity upon both TR and Kk; or, in other words, the semi-vault cannot be sustained by means of the two surfaces TR, and Kk alone.

Let Rk = kR′ = h, Kk = k, and TS = t, being the tangent of the ulterior angle of slope to the radius RS = k. Then the distance of the centre of gravity of the semi-vault KkRT from the middle, Kk, of the key-stone will be

$$= \tfrac{1}{2}h + \frac{3ht+2t^2}{12h+6t}.$$

Farther, we have $t : k :: k :$ DV $= \dfrac{k^2}{t}$.

Therefore KV = KD − DV $= h - \dfrac{k^2}{t} = \dfrac{th-k^2}{t}$.

Hence, to ensure stability, we must have

$$1. \quad \tfrac{1}{2}h + \frac{3ht+2t^2}{12h+6t} > \frac{th-k^2}{t}:$$

Or, taking the limit of tottering equilibrium, we have

$$2. \quad t^3 - 3(h^2 - k^2)\,t + 6hk^2 = 0:$$

from which when two of the three letters are known the third may be found.

Suppose, for example, that a plât-band were constructed upon an equilateral triangle, or such that angle RPR$'$ = 60°. Then TS $= t =$ tan. 30° to rad. Kk. Or, if K$k = k$, be taken $= 1$, then $t =$ tan. $30^\circ = \frac{1}{3}\sqrt{3}$.

$$\text{Hence } \tfrac{1}{27}\,3^{\frac{3}{2}} - (h^2 - 1)\sqrt{3} + 6h = 0.$$

From this equation h, in the case of the limit is found

$$= \tfrac{1}{3}\sqrt{37} + \sqrt{3} = 3{\cdot}7596.$$

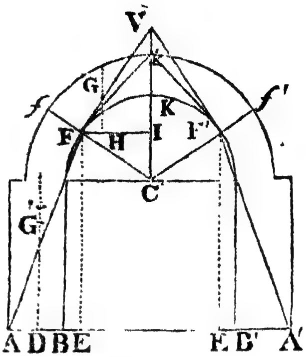
Consequently, in the proposed case, RR$' = 2h$ must be less than 7·5192, or than $7\frac{1}{2}$ times the thickness, Kk, of the key-stone.

Of the Equilibrium of Vaults, regarding the Tenacity of Cements.

120. When the operation of cements is taken into the consideration, the conditions to ensure equilibrium are more easily investigated than when the gravitating tendency of the superincumbent matter is alone regarded. If the cohesive energy of the cement were insuperable, the arch might then be considered as one mass, which would be every where secure, whatever its form might be, provided the piers or abutments were sufficiently strong to resist the horizontal thrust. And, although this property cannot safely be imputed to any cement (strong as many cements are known to be), yet, in a structure, whose component parts are united with a very powerful cement, the matter above an arch will not yield, as when the whole is formed of simple wedges, or as when it would give way in vertical columns, but by the separation of the entire mass into three, or at most, into four pieces: that is, either into the two piers, and the whole mass between them, or into the two piers, and the including mass splitting into two at its crown. It may be advisable, therefore, to investigate the conditions of equilibrium for both these classes of dislocations.

121. Prop. Suppose that the arch Fff'F$'$ tend to fall vertically in one mass, by thrusting out the piers at the joints of fracture, Ff, F$'f'$; it is required to investigate the equations by which the equilibrium may be determined.

Let 2A denote the whole weight of the arch lying between Ff, and F$'f'$, G the centre of gravity of one half of that



arch, the centre of gravity of the whole lying on CV; let P be the weight of one of the piers, reckoned as high as Ff, and G$'$ the place of its centre of gravity.

Now, FV, F$'$V$'$, being respectively perpendicular to Ff, F$'f'$, the weight 2A may be understood to act from V, in the directions VF, VF$'$, and pressing upon the two joints Ff, F$'f'$. The horizontal thrust which it exerts on F, will be

$= a$ tan. FVI $=$ A cot. FCI $= \frac{\text{CI}}{\text{FI}}$: and at the same time the vertical effort will $=$ A.

Now, the first of these forces tends to thrust out the solid AF horizontally, an effort which is resisted by friction; and since it is known that, *cæteris paribus*, the friction varies as the pressure, that is, here, as the weight, we shall have for the resisting force, f . A $+ f$. P. Equating this with the above expression, A . $\frac{\text{CI}}{\text{FI}}$, we obtain for the first equation of equilibrium

$$f \cdot \text{P} = \text{A}\left(\frac{\text{CI}}{\text{FI}} - f\right) \qquad \text{(I.)}$$

Moreover, the horizontal thrust that tends to overturn the pier AF about the angle A, must be regarded as acting at the arm of lever FE, and, therefore, as exerting altogether the energy, A . $\frac{\text{CI}}{\text{FI}}$. FE. This is counteracted by the vertical stress A, operating at the horizontal distance AE, and by the weight, P, acting at the distance AD; DG$'$ being the vertical line passing through the centre of gravity, G$'$, of the pier. Hence we have

$$\frac{\text{CI}}{\text{FI}}\ \text{FE} = \text{A} \cdot \text{AE} + \text{P} \cdot \text{AD};$$

and, after a little reduction, there results for the second equation of equilibrium:

$$\text{P} \cdot \frac{\text{AD}}{\text{FE}} = \text{A}\left(\frac{\text{CI}}{\text{FI}} - \frac{\text{AE}}{\text{FE}}\right) \quad . \quad . \quad . \qquad \text{(II.)}$$

122. Prop. Suppose that each of the two halves kF, kF′, of the arch, tend to turn about the vertex k, removing the points F, and F′: it is required to investigate the conditions of equilibrium in that case.

Referring the weight, A, of the semi-arch from its centre of gravity to the direction of the vertical joint kK, its energy is represented by $A \cdot \frac{FH}{EI}$; and the resulting horizontal thrust at A is, evidently, $A \cdot \frac{FH}{FI} \cdot \frac{FI}{kI} = A \cdot \frac{FH}{kI}$. The vertical stress is $= P + A$; and therefore the friction is represented by $f \cdot P + f \cdot A$. Equating this with the above value of the horizontal thrust, that the pier AF may not move horizontally, we have

$$f \cdot B = A\left(\frac{FH}{kI} - f\right) \quad . \quad . \quad . \quad . \quad (\text{I.})$$

Then, considering the arch and piers as a polygon capable of moving about the angles A, F, k, F′, A′, we must, in order to equilibrium, balance the joint action of P and the semi-arch A at the point F, with the horizontal thrust beforementioned, acting at the arm of lever EF. Thus we shall have $P \cdot AD + A \cdot AE = A \cdot \frac{FH}{kI} \cdot EF$:, from which, after due reduction, there results

$$P \cdot \frac{AD}{EF} = A\left(\frac{FH}{kI} - \frac{AE}{EF}\right) \quad . \quad . \quad . \quad . \quad (\text{II.})$$

123. *Corol.* Hence it will be easy to examine the stability of any arch whose parts are cemented as in the hypotheses of these two propositions. Assume different points such as F, in the arch, for which let the numerical values of the equations (I.) and (II.) be computed. To ensure stability, the first members of those equations, which represent the resistance to motion, must exceed the second members; the weakest points will be those in which the excess of the first above the second member is the least.

If the dimensions of the arch were given, and the thickness of the pier required, the same equations would serve for its determination*.

* The principles adopted in the two last propositions are due to De la Hire, and Coulomb, respectively. For a more com-

DYNAMICS.

124. That department of mechanics which relates to the circumstances and effects of bodies in motion (art. 5.) is of great extent, and of very comprehensive application. A selection of its most interesting topics will here be presented; but numerous other problems which, while they fall within its scope, require the aid of the fluxional analysis, will be solved in the collections at the end of this and of the third volume.

GENERAL LAWS OF MOTION, &c.

125. Prop. The quantity of matter, in all bodies, is in the compound ratio of their magnitudes and densities.

That is, b is as md; where b denotes the body or quantity of matter, m its magnitude, and d its density.

For, by art. 10, in bodies of equal magnitude, the mass or quantity of matter is as the density. But, the densities remaining, the mass is as the magnitude; that is, a double magnitude contains a double quantity of matter, a triple magnitude a triple quantity, and so on. Therefore the mass is in the compound ratio of the magnitude and density.

Corol. 1. In similar bodies, the masses are as the densities and cubes of the diameters, or of any like linear dimensions. —For the magnitudes of bodies are as the cubes of the diameters, &c.

Corol. 2. The masses are as the magnitudes and specific gravities.—For, by art. 10 and 17, the densities of bodies are as the specific gravities.

126. *Scholium.* Hence, if b denote any body, or the quantity of matter in it, m its magnitude, d its density, g its specific gravity, and a its diameter or other dimension; then, $\propto$ (pronounced or named *as*) being the mark for general

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prehensive view of this interesting subject, the student may consult Hutton's Tracts, vol. i., the Appendix to Bossut's Mechanics, and Berard's Treatise on the Statics of Vaults and Domes. The pressure of earth, and the strength of materials, will be treated in a subsequent part of this volume.

proportion, from this proposition and its corollaries we have these general proportions:

$$b \propto md \propto mg \propto a^3d,$$

$$m \propto \frac{b}{d} \propto \frac{b}{g} \propto a^3,$$

$$d \propto \frac{b}{m} \propto g \propto \frac{mg}{a^3},$$

$$a^3 \propto \frac{b}{d} \propto m \propto \frac{mg}{d}.$$

127. Prop. The momentum, or quantity of motion, generated by a single impulse, or any momentary force, is as the generating force.

That is, m is as f; where m denotes the momentum, and f the force.

For every effect is proportional to its adequate cause. So that a double force will impress a double quantity of motion; a triple force, a triple motion; and so on. That is, the motion impressed, is as the motive force which produces it.

128. Prop. The momenta, or quantities of motion, in moving bodies, are in the compound ratio of the masses and velocities.

That is, m is as bv.

For, the motion of any body being made up of the motions of all its parts, if the velocities be equal, the momenta will be as the masses; for a double mass will strike with a double force; a triple mass with a triple force; and so on. Again, when the mass is the same, it will require a double force to move it with a double velocity, a triple force with a triple velocity, and so on; that is, the motive force is as the velocity; but the momentum impressed, is as the force which produces it, by art. 127; and therefore the momentum is as the velocity when the mass is the same. But the momentum was found to be as the mass when the velocity is the same. Consequently, when neither are the same, the momentum is in the compound ratio of both the mass and velocity.

Otherwise: $\mathrm{M} : m :: \mathrm{B} : b$, when V is constant:
and $m : \mu :: \mathrm{V} : v$; when B is constant:
therefore, $\mathrm{M} : \mu :: \mathrm{BV} : bv$, when both vary.

129. Prop. In uniform motions, the spaces described are in the compound ratio of the velocities and the times of their description.

That is, s is as tv.

For, by the nature of uniform motion,

S T : t, when V is constant:
and s V : v, when T is constant:
therefore S σ TV : tv, when both vary.

Corol. 1. In uniform motions, the time is as the space directly, and velocity reciprocally; or as the space divided by the velocity. And when the velocity is the same, the time is as the space. But when the space is the same, the time is reciprocally as the velocity.

Corol. 2. The velocity is as the space directly and the time reciprocally; or as the space divided by the time. And when the time is the same, the velocity is as the space. But when the space is the same, the velocity is reciprocally as the time.

Scholium.

130. In uniform motions generated by momentary impulse, let b = any body or quantity of matter to be moved,
f = force of impulse acting on the body b,
v = the uniform velocity generated in b,
m = the momentum generated in b,
s = the space described by the body b,
t = the time of describing the space s with the veloc. v.

Then from the last three propositions and corollaries, we have these three general proportions, namely, $f \propto m$, $m \propto bv$, and $s \propto tv$; from which is derived the following table of the general relations of those six quantities, in uniform motions, and impulsive or percussive forces:

$$f \propto m \propto bv \propto \frac{bs}{t}.$$

$$m \propto f \propto bv \propto \frac{bs}{t}.$$

$$b \propto \frac{f}{v} \propto \frac{m}{v} \propto \frac{ft}{s} \propto \frac{mt}{s}.$$

$$s \propto tv \propto \frac{ft}{b} \propto \frac{tm}{b}.$$

$$v \propto \frac{s}{t} \propto \frac{f}{b} \propto \frac{m}{b}.$$

$$t \propto \frac{s}{v} \propto \frac{bs}{f} \propto \frac{bs}{m}.$$

By means of which, may be resolved all questions relating

to uniform motions, and the effects of momentary or impulsive forces.

131. Prop. The momentum generated by a constant and uniform force, acting for any time, is in the compound ratio of the force and time of acting.

That is, m is as ft.

For, supposing the time divided into very small parts, by art. 127, the momentum in each particle of time is the same, and therefore the whole momentum will be as the whole time, or sum of all the small parts. But by the same prop. the momentum for each small time, is also as the motive force. Consequently the whole momentum generated, is in the compound ratio of the force and time of acting.

Corol. 1. The motion, or momentum, lost or destroyed in any time, is also in the compound ratio of the force and time. For whatever momentum any force generates in a given time; the same momentum will an equal force destroy in the same or equal time; acting in a contrary direction.

And the same is true of the increase or decrease of motion, by forces that conspire with, or oppose the motion of bodies.

Corol. 2. The velocity generated, or destroyed, in any time, is directly as the force and time, and reciprocally as the body or mass of matter.—For, by this and art. 128, the compound ratio of the body and velocity, is as that of the force and time; and therefore the velocity is as the force and time divided by the body. And if the body and force be given, or constant, the velocity will be as the time.

132. Prop. The spaces passed over by bodies, urged by any constant and uniform forces, acting during any times, are in the compound ratio of the forces and squares of the times directly, and the body or mass reciprocally.

Or, the spaces are as the squares of the times, when the force and body are given.

That is, s is as $\frac{ft^2}{b}$, or as t^2 when f and b are given. For, let v denote the velocity acquired at the end of any time t, by any given body b, when it has passed over the space s. Then, because the velocity is as the time, by the last corol. therefore $\frac{1}{2}v$ is the velocity at $\frac{1}{2}t$, or at the middle point of the time; and as the increase of velocity is uniform, the same space s will be described in the same time t, by the velocity $\frac{1}{2}v$ uniformly continued from beginning to end. But, in uniform motions, the space is in the compound ratio

of the time and velocity; therefore s is as $\frac{1}{2}tv$, or indeed $s = \frac{1}{2}tv$. But, by the last corol. the velocity v is as $\frac{ft}{b}$, or as the force and time directly, and as the body reciprocally. Therefore s, or $\frac{1}{2}tv$, is as $\frac{ft^2}{b}$; that is, the space is as the force and square of the time directly, and as the body reciprocally. Or s is as t^2, the square of the time only, when b and f are given.

Corol. 1. The space s is also as tv, or in the compound ratio of the time and velocity; b and f being given. For, $s = \frac{1}{2}tv$ is the space actually described. But tv is the space which might be described in the same time t, with the last velocity v, if it were uniformly continued for the same or an equal time. Therefore the space s, or $\frac{1}{2}tv$, which is actually described, is just half the space tv, which would be described with the last or greatest velocity, uniformly continued for an equal time t.

Corol. 2. The space s is also as v^2, the square of the velocity; because the velocity v is as the time t.

Scholium.

133. The last four propositions give theorems for resolving all questions relating to motions uniformly accelerated. Thus, put b = any body or quantity of matter,
f = the force constantly acting on it,
t = the time of its acting,
v = the velocity generated in the time t,
s = the space described in that time,
m = the momentum at the end of the time.

Then, from these fundamental relations, $m \propto bv$, $m \propto ft$, $s \propto tv$, and $v \propto \frac{ft}{b}$, we obtain the following table of the general relations of uniformly accelerated motions:

$$m \propto bv \propto ft \propto \frac{bs}{t} \propto \frac{fs}{v} \propto \frac{ft^2v}{s} \propto \sqrt{bfs} \propto \sqrt{bftv}.$$

$$b \propto \frac{m}{v} \propto \frac{ft}{v} \propto \frac{mt}{s} \propto \frac{ft^2}{s} \propto \frac{f^2t^3}{ms} \propto \frac{m^2}{fs} \propto \frac{m^2}{ftv} \propto \frac{fs}{v^2}.$$

$$f \propto \frac{m}{t} \propto \frac{bv}{t} \propto \frac{mv}{s} \propto \frac{ms}{t^2v} \propto \frac{m^2}{bs} \propto \frac{m^2}{btv} \propto \frac{bv^2}{s} \propto \frac{bs}{t^2}.$$

$$v \propto \frac{s}{t} \propto \frac{ft}{b} \propto \frac{m}{b} \propto \frac{ms}{ft^2} \propto \frac{fs}{m} \propto \frac{m^2}{bft} \propto \sqrt{\frac{fs}{b}} \propto \frac{f^2st}{m^2}.$$

$$s \propto tv \propto \frac{ft^2}{b} \propto \frac{mt}{b} \propto \frac{ft^2v}{m} \propto \frac{mv}{f} \propto \frac{m^2}{bf} \propto \frac{bv^2}{f} \propto \frac{m^2v}{f^2t}.$$

$$t \propto \frac{s}{v} \propto \frac{m}{f} \propto \frac{bv}{f} \propto \frac{bs}{m} \propto \sqrt{\frac{bs}{f}} \propto \sqrt{\frac{ms}{fv}} \propto \frac{m^2}{bfv}, \text{ \&c.}$$

134. From the above relations those quantities are to be left out which are given, or which are proportional to each other. Thus, if the body or quantity of matter be always the same, then the space described is as the force and square of the time. And if the body be proportional to the force, as all bodies are in respect to their gravity; then the space described is as the square of the time, or square of the velocity; and in this case, if F be put $= \frac{f}{b}$, the accelerating force; then will

$$s \propto tv \propto Ft^2 \propto \frac{\quad}{F}.$$

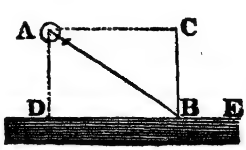
$$v \propto \frac{\quad}{\quad} \propto Ft \propto \sqrt{Fs}.$$

$$t \propto \frac{\quad}{\quad} \propto \frac{v}{F} \propto \sqrt{\frac{s}{F}}.$$

ON THE COLLISION OF BODIES.

135. Prop. If a body strike or act obliquely on a plain surface, the force or energy of the stroke, or action, is as the sine of the angle of incidence.

Or, the force on the surface is to the same if it had acted perpendicularly, as the sine of incidence is to radius.



Let AB express the direction and the absolute quantity of the oblique force on the plane DE; or let a given body A, moving with a certain velocity, impinge on the plane at B; then its force will be to the action on the plane, as radius to the sine of the angle ABD, or as AB to AD or BC, drawing AD and BC perpendicular, and AC parallel to DE.

For, by art. 29, the force AB is equivalent to the two forces AC, CB; of which the former AC does not act on the

plane, because it is parallel to it. The plane is therefore only acted on by the direct force CB, which is to AB, as the sine of the angle BAC, or ABD, to radius.

Corol. 1. If a body act on another, in any direction, and by any kind of force, the action of that force on the second body, is made only in a direction perpendicular to the surface on which it acts. For the force in AB acts on DE only by the force CB, and in that direction.

Corol. 2. If the plane DE be not absolutely fixed, it will move, after the stroke, in the direction perpendicular to its surface. For it is in that direction that the force is exerted.

136. PROP. If one body A, strike another body B, which is either at rest or moving towards the body A, or moving from it, but with a less velocity than that of A; then the momenta, or quantities of motion, of the two bodies, estimated in any one direction, will be the very same after the stroke that they were before it.

For, because action and re-action are always equal, and in contrary directions, art. 20, whatever momentum the one body gains one way by the stroke, the other must just lose as much in the same direction; and therefore the quantity of motion in that direction, resulting from the motions of both the bodies, remains still the same as it was before the stroke.

137. Thus, if A with a momentum of 10, strike B at rest, and communicate to it a momentum of 4, in the direction AB. Then A will have only a momentum of 6 in that direction; which, together with the momentum of B, viz. 4, make up still the same momentum between them as before, namely 10.

138. If B were in motion before the stroke, with a momentum of 5, in the same direction, and receive from A an additional momentum of 2. Then the motion of A after the stroke will be 8, and that of B, 7; which between them make 15, the same as 10 and 5, the motions before the stroke.

139. Lastly, if the bodies move in opposite directions, and meet one another, namely, A with a motion of 10, and B, of 5; and A communicate to B a motion of 6 in the direction AB of its motion. Then, before the stroke, the whole motion from both, in the direction of AB, is 10 − 5 or 5. But, after the stroke, the motion of A is 4 in the direction AB, and the motion of B is 6 − 5 or 1 in the same direction AB;

therefore the sum $4 + 1$, or 5, is still the same motion from both, as it was before.

140. Prop. The motion of bodies included in a given space, is the same with regard to each other, whether that space be at rest, or move uniformly in a right line.

For, if any force be equally impressed both on the body and the line in which it moves, this will cause no change in the motion of the body along the right line. For the same reason, the motions of all the other bodies, in their several directions, will still remain the same. Consequently their motions among themselves will continue the same, whether the including space be at rest, or be moved uniformly forward. And therefore their mutual actions on one another, must also remain the same in both cases.

141. Prop. If a hard and fixed plane be struck by either a soft or a hard unelastic body, the body will adhere to it. But if the plane be struck by a perfectly elastic body, it will rebound from it again with the same velocity with which it struck the plane.

For, since the parts which are struck, of the elastic body, suddenly yield and give way by the force of the blow, and as suddenly restore themselves again with a force equal to the force which impressed them, by the definition of elastic bodies; the intensity of the action of that restoring force on the plane, will be equal to the force or momentum with which the body struck the plane. And, as action and re-action are equal and contrary, the plane will act with the same force on the body, and so cause it to rebound or move back again with the same velocity as it had before the stroke.

But hard or soft bodies, being devoid of elasticity, by the definition, having no restoring force to throw them off again, they must necessarily adhere to the plane struck.

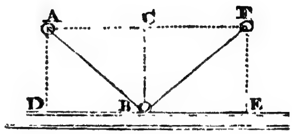
142. *Corol.* 1. The effect of the blow of the elastic body, on the plane, is double to that of the unelastic one, the velocity and mass being equal in each.

For the force of the blow from the unelastic body, is as its mass and velocity, which is only destroyed by the resistance of the plane. But in the elastic body, that force is not only destroyed and sustained by the plane; but another also equal to it is sustained by the plane, in consequence of the restoring force, and by virtue of which the body is thrown back again with an equal velocity. And therefore the intensity of the blow is doubled.

143. *Corol.* 2. Hence unelastic bodies lose, by their col-

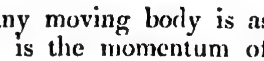
lision, only half the motion lost by elastic bodies; their mass and velocities being equal.—For the latter communicate double the motion of the former.

144. Prop. If an elastic body A impinge on a firm plane DE at the point B, it will rebound from it in an angle equal to that in which it struck it; or the angle of incidence will be equal to the angle of reflection; namely, the angle ABD equal to the angle FBE.



Let AB express the force of the body A in the direction AB; which let be resolved into the two AC, CB, parallel and perpendicular to the plane.—Take BE and CF equal to AC, and draw BF. Now action and re-action being equal, the plane will resist the direct force CB by another BC equal to it, and in a contrary direction; whereas the other AC, being parallel to the plane, is not acted on or diminished by it, but still continues as before. The body is therefore reflected from the plane by two forces BC, BE, perpendicular and parallel to the plane, and therefore moves in the diagonal BF by composition. But, because AC is equal to BE or CF, and that BC is common, the two triangles BCA, BCF are mutually similar and equal; and consequently the angles at A and F are equal, as also their equal alternate angles ABD, FBE, which are the angles of incidence and reflection.

145. Prop. To determine the motion of non-elastic bodies, when they strike each other directly, or in the same line of direction.



Let the non-elastic body B, moving with the velocity V in the direction Bb, and the body b with the velocity v, strike each other. Then, because the momentum of any moving body is as the mass into the velocity, BV $=$ M is the momentum of the body B, and $bv = m$ the momentum of the body b, which let be the less powerful of the two motions. Then, by art. 136, the bodies will both move together as one mass in the direction BC after the stroke, whether before the stroke the body b moved towards C or towards B. Now, according as that motion of b was from or towards B, that is, whether the motions were in the same or contrary ways, the momentum after the stroke, in direction BC, will be the sum or difference of the momentums before the stroke; namely, the momentum in direction BC will be

$BV + bv$, if the bodies moved the same way, or
$BV - bv$, if they moved contrary ways, and
BV only, if the body b were at rest.

Then divide each momentum by the common mass of matter $B + b$, and the quotient will be the common velocity after the stroke in the direction BC; namely, the common velocity will be, in the first case,

$$\frac{BV + bv}{B + b}, \text{ in the 2d } \frac{BV - bv}{B + b}, \text{ and in the 3d } \frac{BV}{B + b}.$$

Corol. $V - \frac{BV + bv}{B + b} = \frac{V - v}{B + b} \times b$, the veloc. lost by B.

146. For example, if the bodies, or weights, B and b, be as 5 to 3, and their velocities V and v, as 6 to 4, or as 3 to 2, before the stroke; then 15 and 6 will be as their momentums, and 8 the sum of their weights; consequently, after the stroke, the common velocity will be as

$$\frac{15 + 6}{8} = \frac{21}{8} \text{ or } 2\tfrac{5}{8} \text{ in the first case,}$$

$$\frac{15 - 6}{8} = \frac{9}{8} \text{ or } 1\tfrac{1}{8} \text{ in the second, and}$$

$$\frac{15}{8} \text{ - - - or } 1\tfrac{7}{8} \text{ in the third.}$$

147. PROP. If two perfectly elastic bodies impinge on one another, their relative velocity will be the same both before and after the impulse: that is, they will recede from each other with the same velocity with which they approached and met.

For the compressing force is as the intensity of the stroke; which, in given bodies, is as the relative velocity with which they meet or strike. But perfectly elastic bodies restore themselves to their former figure, by the same force by which they were compressed; that is, the restoring force is equal to the compressing force, or to the force with which the bodies approach each other before the impulse. But the bodies are impelled from each other by this restoring force; and therefore this force, acting on the same bodies, will produce a relative velocity equal to that which they had before: or it will make the bodies recede from each other with the same velocity with which they before approached, or so as to be equally distant from one another at equal times before and after the impact.

148. *Remark.* It is not meant by this proposition, that each body will have the same velocity after the impulse as it had before; for that will be varied according to the relation of the masses of the two bodies; but that the velocity of the one will be, after the stroke, so much increased, and the other decreased, as to have the same difference as before, in one and the same direction. So, if the elastic body B move with a velocity V, and overtake the elastic body b moving the same way with the velocity v; then their relative velocity, or that with which they strike, is $V - v$, and it is with this same velocity that they separate from each other after the stroke. But if they meet each other, or the body b move contrary to the body B; then they meet and strike with the velocity $V + v$, and it is with the same velocity that they separate and recede from each other after the stroke. But whether they move forward or backward after the impulse, and with what particular velocities, are circumstances that depend on the various masses and velocities of the bodies before the stroke, and which make the subject of the next proposition.—It may further be remarked, that the sums of the two velocities, of each body, before and after the stroke, are equal to each other. Thus, V, v being the velocities before the impact, if x and y be the corresponding ones after it; since $V - v = y - x$, therefore $V + x = v + y$.

149. PROP. To determine the motions of elastic bodies after striking each other directly.

Let the elastic body B move in the direction BC, with the velocity V; and let the velocity of the other body b be v in the same line; which latter velocity v will be positive if b move the same way as B, but negative if b move in the opposite direction to B. Then their relative velocity in the direction BC is $V - v$; also the momenta before the stroke are BV and bv, the sum of which is $BV + bv$ in the direction BC.

Again, put x for the velocity of B, and y for that of b, in the same direction BC, after the stroke; then their relative velocity is $y - x$, and the sum of their momenta $Bx + by$ in the same direction.

But the momenta before and after the collision, estimated in the same direction, are equal, by art. 136, as also the relative velocities, by the last prop. Whence arise these two equations:

$$\text{viz. } BV + bv = Bx + by,$$
$$\text{and } V - v = y - x;$$

the resolution of which equations gives

$$x = \frac{(B - b)V + 2bv}{B + b}, \text{ the velocity of B,}$$

$$y = \frac{-(B - b)v + 2BV}{B + b}, \text{ the velocity of } b.$$

Or, $x = V - \frac{2b}{B + b}(V - v)$, and $y = v + \frac{2B}{B + b}(V - v)$.

So that the velocity lost by B is $\frac{2b}{B + b}(V - v)$,

and the velocity gained by b is $\frac{2B}{B + b}(V - v)$;

which two velocities are in the ratio of b to B, or reciprocally as the two bodies themselves.

Corol. 1. The velocity lost by B drawn into B, and the velocity gained by b drawn into b, give each of them $\frac{2Bb}{B + b}(V - v)$, for the momentum gained by the one and lost by the other, by the stroke; which increment and decrement being equal, they cancel one another, and leave the same momentum $BV + bv$ after the impact, as it was before it.

Corol. 2. Hence also, $BV^2 + bv^2 = Bx^2 + by^2$, or the sum of the vires vivarum is always preserved the same, both before and after the impact. For, since

$$BV + bv = Bx + by,$$
$$\text{or } BV - Bx = by - bv,$$
$$\text{and } V + x = y + v, \text{ these two equas. multiplied,}$$
$$\text{give } BV^2 - Bx^2 = by^2 - bv^2,$$
$$\text{or } BV^2 + bv^2 = Bx^2 + by^2,$$

the equation of the so called living forces.

Corol. 3. But if v be negative, or the body b moved in the contrary direction before collision, or towards B; then, changing the sign of v, the same theorems become

$$x = \frac{(B - b)V - 2bv}{B + b}, \text{ the velocity of B,}$$

$$y = \frac{(B - b)v + 2BV}{B + b}, \text{ the veloc. of } b\text{, in the direction BC.}$$

And if b were at rest before the impact, making its velocity $v = 0$, the same theorems give

$$x = \frac{B - b}{B + b}V, \text{ and } y = \frac{2B}{B + b}V, \text{ the velocities in this case.}$$

And, in this case, if the two bodies B and b be equal to each other; then $B - b = 0$, and $\frac{2B}{B + b} = \frac{2B}{2B} = 1$; which give $x = 0$, and $y = V$; that is, the body B will stand still, and the other body b will move on with the whole velocity of the former; a thing which we sometimes see happen in playing at billiards; and which would happen much oftener if the balls were perfectly elastic.

Scholium.

150. If the bodies be elastic only in a partial degree, the sum of the momenta will still be the same, both before and after collision; but the velocities after, will be less than in the case of perfect elasticity, in the ratio of the imperfection. Hence, with the same notation as before, the two equations will now be $BV + bv = Bx + by$,

$$\text{and } V - v = \frac{m}{n}(y - x),$$

where m to n denotes the ratio of perfect to imperfect elasticity. And the resolution of these two equations, gives the following values of x and y, viz.

$$x = V - \frac{m + n}{m} \cdot \frac{b}{B + b}(V - v),$$

$$y = v + \frac{m + n}{m} \cdot \frac{B}{B + b}(V - v),$$

for the velocities of the two bodies after impact in the case of imperfect elasticity: and these would become the same as the former if n were $= m$.

Hence, if the two bodies B and b be equal, then

$$x = V - \frac{m + n}{2m}(V - v), \text{ and } y = v + \frac{m + n}{2m}(V - v),$$

where the velocity lost by B is just equal to that gained by b. And if in this case b was at rest before the impact, or $v = 0$, then the resulting motions would be

$$x = \frac{m - n}{2m}V, \text{ and } y = \frac{m + n}{2m}V,$$

which are in the ratio of $m - n$ to $m + n$.

Also, if $m = n$, or the bodies perfectly elastic, then $x = 0$, and $y = V$; or B would be at rest, and b go on with the first motion of B.

Further, in this case also, the velocity of B before the impact, is to that of b after it, as V to $\frac{m + n}{2m}V$, or as $2m$ to

$m+n$. But, if the bodies be now supposed to vibrate in circles, as pendulums, in which case the chords (C and c) of the arcs described are known to be proportional to the velocities; then it will be $2m : m+n :: C : c$; hence $m : n :: C : 2c - C$. So that, by measuring these chords, of the arcs thus experimentally described, the ratio of m to n, or the degree of elasticity in the bodies, may be determined.

151. PROP. The greatest velocity which can be generated by the propagation of motion through a row of contiguous perfectly elastic bodies, will be when those bodies are in geometrical progression.

First, take three bodies, A, X, and C: then (art. 149) the velocity communicated from A to X $= \frac{2Aa}{A+X}$, a being the velocity of A: and when the body X impinges upon C at rest with this velocity, the vel. communicated to C will

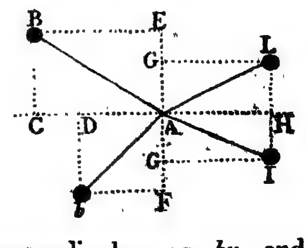
$$= \frac{2X}{X+C} \cdot \frac{2Aa}{A+X} = \frac{4AaX}{(A+X)(X+C)}$$

$$= \frac{4Aa}{(A \div X + 1)(X+C)} = \frac{4Aa}{A+X+(AC \div X)+C}.$$

This fraction is evidently a max. when its denominator is a min. that is, since A and C are given, when $X^2 = AC$, or when X is a mean proportional between A and C.

For the same reason the velocity communicated from the second body through C the third, to a fourth, D, will be greatest when C is a mean proportional between the second and fourth. Like reasoning will evidently hold for a series of perfectly elastic bodies. Farther, if the number of bodies in the geometrical progression be increased without limit, the quantity of motion communicated to the last, from a given quantity of motion in the first, however small, may also be increased without limit.

152. PROP. If bodies strike one another obliquely, it is proposed to determine their motions after the stroke.



Let the two bodies B, b, move in the oblique directions BA, bA, and strike each other at A, with velocities which are in proportion to the lines BA, bA; to find their motions after the impact. Let CAH represent the plane in which the bodies touch in the point of concourse; to which draw the perpendiculars BC, bD, and

complete the rectangles CE, DF. Then the motion in BA is resolved into the two BC, CA; and the motion in *b*A is resolved into the two *b*D, DA; of which the antecedents BC, *b*D, are the velocities with which they directly meet, and the consequents CA, DA, are parallel; therefore by these the bodies do not impinge on each other, and consequently the motions, according to these directions, will not be changed by the impulse; so that the velocities with which the bodies meet, are as BC and *b*D, or their equals EA and FA. The motions therefore of the bodies B, *b*, directly striking each other with the velocities EA, FA, will be determined by art. 145 or 149, according as the bodies are elastic or non-elastic; which being done, let AG be the velocity, so determined, of one of them, as A; and since there remains also in the body a force of moving in the direction parallel to BE, with a velocity as BE, make AH equal to BE, and complete the rectangle GH: then the two motions in AH and AG, or HI, are compounded into the diagonal AI, which therefore will be the path and velocity of the body B after the stroke. And after the same manner is the motion of the other body *b* determined after the impact.

If the elasticity of the bodies be imperfect in any given degree, then the quantity of the corresponding lines must be diminished in the same proportion. For the full consideration of this branch of the inquiry the student is referred to the Treatises of Mechanics by *Gregory* and *Bridge*.

Problems for Exercise on Collision.

EXAM. 1. A cannon ball weighing 12lbs. moving with a velocity of 1200 feet per second, *meets* another of 18lbs. moving with a velocity of 1000 feet per second. Required the velocity of each after impact, supposing both to be non-elastic.

EXAM. 2. B and *b* are as 3 to 2, and the velocity of B is to that of *b* as 5 to 4. They are perfectly hard, and move before impact in the same direction; what are the velocities lost by B and gained by *b*?

EXAM. 3. B and *b* are perfectly elastic, and move in opposite directions. B is triple of *b*, but *b*'s velocity is double that of B. How do those bodies move after impact?

EXAM. 4. A body whose elasticity is to perfect elasticity as 15 to 16, falls from the height of 100 feet upon a perfectly hard horizontal plane. It then rebounds and falls again, and so on, always in a vertical direction. It is required to find the whole space described by the body

before its motion ceases, as well as the entire time of its motion.

EXAM. 5. Investigate what must be the force of elasticity, so that the sums of the products formed by multiplying each body into *any* assumed power, n, of its velocity, may not be altered by the impact of the two bodies.

THE LAWS OF GRAVITY; THE DESCENT OF HEAVY BODIES; AND THE MOTION OF PROJECTILES IN FREE SPACE.

153. PROP. ALL the properties of motion delivered in art. 132, its corollaries and scholium, for constant forces, are true in the motions of bodies freely descending by their own gravity; namely, that the velocities are as the times, and the spaces as the squares of the times, or as the squares of the velocities.

For, since the force of gravity is uniform, and constantly the same, at all places near the earth's surface, or at nearly the same distance from the centre of the earth; and since this is the force by which bodies descend to the surface; they therefore descend by a force which acts constantly and equally; consequently all the motions freely produced by gravity, are as above specified, by that proposition, &c.

SCHOLIUM.

154. Now it has been found, by numberless experiments, that gravity is a force of such a nature, that all bodies, whether light or heavy, fall vertically through equal spaces in the same time, abstracting from the resistance of the air; as lead or gold and a feather, which in an exhausted receiver fall from the top to the bottom in the same time. It is also found that the velocities acquired by descending, are in the exact proportion of the times of descent: and further, that the spaces descended are proportional to the squares of the times, and therefore to the squares of the velocities. Hence then it follows, that the weights or gravities, of bodies near the surface of the earth, are proportional to the quantities of matter contained in them; and that the spaces, times, and velocities, generated by gravity, have the relations contained in the three general proportions before laid down. Further, as it is found, by accurate experiments, that a body in the

latitude of London, falls nearly $16\frac{1}{12}$ feet in the first second of time, and consequently that at the end of that time it has acquired a velocity double, or of $32\frac{1}{6}$ feet by corol. 1, art. 132; therefore, if $\frac{1}{2}g$ denote $16\frac{1}{12}$ feet, the space fallen through in one second of time, or g the velocity generated in that time; then, because the velocities are directly proportional to the times, and the spaces to the squares of the times; therefore it will be,

$$\text{as } 1'' : t'' :: g : gt = v \text{ the velocity,}$$
$$\text{and } 1^2 : t^2 :: \tfrac{1}{2}g : \tfrac{1}{2}gt^2 = s \text{ the space.}$$

So that, for the descents of gravity, we have these general equations, namely,

$$= \tfrac{1}{2}gt^2 = \frac{v^2}{2g} = \tfrac{1}{2}tv.$$
$$= gt = \frac{2s}{t} = \sqrt{2gs}$$
$$= \frac{v}{g} = \frac{2s}{v} = \sqrt{\frac{2s}{g}}.$$
$$g = \frac{v}{t} = \frac{2s}{t^2} = \frac{v^2}{2s}.$$

Hence, because the times are as the velocities, and the spaces as the squares of either, therefore,

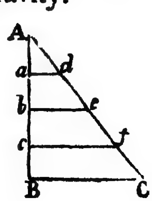
if the times be as the numbs.	1,	2,	3,	4,	5,	&c.
the velocities will also be as	1,	2,	3,	4,	5,	&c.
and the spaces as their squares	1,	4,	9,	16,	25,	&c.
and the space for each time as	1,	3,	5,	7,	9,	&c.

namely, as the series of the odd numbers, which are the differences of the squares denoting the whole spaces. So that if the first series of natural numbers be seconds of time,

namely, the times in seconds	$1''$,	$2''$,	$3''$,	$4''$,	&c.
the velocities in feet will be	$32\frac{1}{6}$,	$64\frac{1}{3}$,	$96\frac{1}{2}$,	$128\frac{2}{3}$,	&c.
the spaces in the whole times	$16\frac{1}{12}$,	$64\frac{1}{3}$,	$144\frac{3}{4}$,	$257\frac{1}{3}$,	&c.
and the space for each second	$16\frac{1}{12}$,	$48\frac{1}{4}$,	$80\frac{5}{12}$,	$112\frac{7}{12}$,	&c.

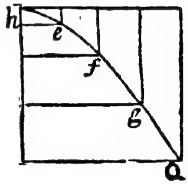
of which spaces the common difference is $32\frac{1}{6}$ feet, the natural and obvious measure of g, the force of gravity.

155. These relations, of the times, velocities, and spaces, may be represented by certain lines and geometrical figures. Thus, if the line AB denote the time of any body's descent, and BC, at right angles to it, the velocity gained at the end of that time; by joining AC, and dividing the time AB into any number of parts at the points a, b, c;



then shall *ad, be, cf,* parallel to BC, be the velocities at the points of time *a, b, c,* or at the ends of the times, A*a*, A*b*, A*c*; because these latter lines, by similar triangles, are proportional to the former *ad, be, cf,* and the times are proportional to the velocities. Also, the area of the triangle ABC will represent the space descended by the force of gravity in the time AB, in which it generates the velocity BC; because that area is equal to $\frac{1}{2}$AB × BC, and the space descended is $s = \frac{1}{2}tv$, or half the product of the time and the last velocity. And, for the same reason, the less triangles A*ad*, A*be*, A*cf*, will represent the several spaces described in the corresponding times A*a*, A*b*, A*c*, and velocities *ad, be, cf*; those triangles or spaces being also as the squares of their like sides A*a*, A*b*, A*c*, which represent the times, or of *ad, be, cf,* which represent the velocities.

156. But as areas are rather unnatural representations of the spaces passed over by a body in motion, which are lines, the relations may better be represented by the abscisses and ordinates of a parabola. Thus, if PQ be a parabola, PR its axis, and RQ its ordinate; and P*a*, P*b*, P*c*, &c, parallel to RQ, represent the times from the beginning, or the velocities, then *ae, bf, cg,* &c, parallel to the axis PR, will represent the spaces described by a falling body in those times; for, in a parabola, the abscisses P*h*, P*i*, P*k*, &c, or *ae, bf, cg,* &c, which are the spaces described, are as the squares of the ordinates *he, if, kg,* &c, or P*a*, P*b*, P*c*, &c, which represent the times or velocities.



157. And because the laws for the destruction of motion, are the same as those for the generation of it, by equal forces, but acting in a contrary direction; therefore,

1*st*, A body thrown directly upward, with any velocity, will lose equal velocities in equal times.

2*d*, If a body be projected upward, with the velocity it acquired in any time by descending freely, it will lose all its velocity in an equal time, and will ascend just to the same height from which it fell, and will describe equal spaces in equal times, in rising and falling, but in an inverse order; and it will have equal velocities at any one and the same point of the line described, both in ascending and descending.

3*d*, If bodies be projected upward, with any velocities, the height ascended to, will be as the squares of those velocities, or as the squares of the times of ascending, till they lose all their velocities.

158. In solving problems, where a body, instead of being permitted to fall freely, is projected vertically upwards or downwards with a given velocity, it will assist the comprehension of what takes place, to ascertain what results from the original projection, and what from the force of gravity. Thus, if a body be projected with a velocity v it will, in the time t, describe the space tv (art. 129) apart from the operation of gravity or any other force. Blending this with the preceding expression for the space described by a falling body, we have

$$s = tv \mp \tfrac{1}{2} g t^2,$$

in which the *lower* sign must be employed when the projection is vertically *downwards*, the *upper* when the projection is vertically *upwards*.

EXERCISES ON RISING AND FALLING BODIES.

1. Find the space descended vertically by a body in 7 seconds of time, and the velocity acquired?

Ans. $788\frac{1}{12}$, space; $225\frac{1}{6}$, velocity.

2. Required the time of generating a velocity of 100 feet per second, and the whole space descended.

Ans. $3''\frac{21}{193}$, time; $155\frac{85}{193}$ *f.* space.

3. Find the time of descending 400 feet, and the velocity at the end of that time.

Ans. $4''\frac{76}{77}$, time; $160\frac{32}{77}$, velocity.

4. If a body fall freely for $5''$, how far will it descend during the last second of its motion?

5. If an arrow be propelled vertically upwards from a bow with a velocity of $96\frac{1}{2}$ feet per second, how high will it rise, and how long will it be before it returns again to the ground?

6. If a ball be projected vertically *downwards* with a velocity of 100 feet per second, how far will it have descended in three seconds?

7. If a ball be projected *upwards* with a velocity of 100 feet per second, how far will it have arisen in three seconds?

8. If a ball be projected vertically upwards with a velocity of 44 feet per second, will it be above or below the point of projection in four seconds, the force of gravity tending all the time to draw it downwards?

9. A drop of rain falls through $176\frac{1}{12}$ feet in the last second; how high is the cloud from which it descended?

10. A body falling freely was observed to pass through

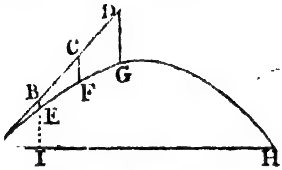
half its descent in the last second; how far did it fall, and how long was it in falling?

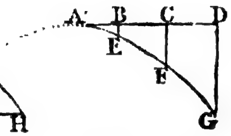
11. Two weights, one of 5lbs. the other of 3lbs. hang freely over a pulley: after motion is allowed to commence how far will the larger weight descend, or the smaller arise, in four seconds?

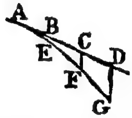
N. B. The theorem for operation is $s = \frac{W-w}{W+w} \cdot \frac{1}{2}gt^2$.

12. Two equal weights are balanced over a pulley. A pound weight being added to one of them, and motion in consequence taking place, the preponderating weight descended through $16\frac{1}{12}$ feet in four seconds. Required the measure of the two equal weights?

158. Prop. If a body be projected in free space, either parallel to the horizon, or in an oblique direction, by the force of gunpowder, or any other impulse; it will, by this motion, in conjunction with the action of gravity, describe the curve line of a parabola.







Let the body be projected from the point A, in the direction AD, with any uniform velocity: then, in any equal portions of time, it would, by art. 129, describe the equal spaces AB, BC, CD, &c, in the line AD, if it were not drawn continually down below that line by the action of gravity. Draw BE, CF, DG, &c, in the direction of gravity, or perpendicular to the horizon, and equal to the spaces through which the body would descend by its gravity in the same time in which it would uniformly pass over the corresponding spaces AB, AC, AD, &c, by the projectile motion. Then, since by these two motions the body is carried over the space AB, in the same time as over the space BE, and the space AC in the same time as the space CF, and the space AD in the same time as the space DG, &c; therefore, by the composition of motions, at the end of those times, the body will be found respectively in the points E, F, G, &c; and consequently the real path of the projectile will be the curve line AEFG &c. But the spaces AB, AC, AD, &c, described by uniform motion, are as the times of description; and the

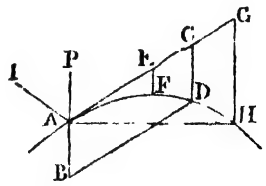
spaces BE, CF, DG, &c, described in the same times by the accelerating force of gravity, are as the squares of the times; consequently the perpendicular descents are as the squares of the spaces in AD, that is BE, CF, DG, &c, are respectively proportional to AB^2, AC^2, AD^2, &c; which is the property of the parabola by theor. 8, Con. Sect. Therefore the path of the projectile is the parabolic line AEFG &c, to which AD is a tangent at the point A.

159. *Corol.* 1. The horizontal velocity of a projectile, is always the same constant quantity, in every point of the curve; because the horizontal motion is in a constant ratio to the motion in AD, which is the uniform projectile motion. And the projectile velocity is in proportion to the constant horizontal velocity, as radius to the cosine of the angle DAH, or angle of elevation or depression of the piece above or below the horizontal line AH.

160. *Corol.* 2. The velocity of the projectile in the direction of the curve, or of its tangent at any point A, is as the secant of its angle BAI of direction above the horizon. For the motion in the horizontal direction AI is constant, and AI is to AB, as radius to the secant of the angle A; therefore the motion at A, in AB, is everywhere as the secant of the angle A.

161. *Corol.* 3. The velocity in the direction DG of gravity, or perpendicular to the horizon, at any point G of the curve, is to the first uniform projectile velocity at A, or point of contact of a tangent, as 2GD is to AD. For, the times in AD and DG being equal, and the velocity acquired by freely descending through DG, being such as would carry the body uniformly over twice DG in an equal time, and the spaces described with uniform motions being as the velocities, therefore the space AD is to the space 2DG, as the projectile velocity at A, to the perpendicular velocity at G.

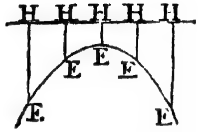
162. PROP. The velocity in the direction of the curve, at any point of it, as A, is equal to that which is generated by gravity in freely descending through a space which is equal to one-fourth of the parameter of the diameter of the parabola at that point.



Let PA or AB be the height due to the velocity of the projectile at any point A, in the direction of the curve or tangent AC, or the velocity acquired by falling through that height; and complete the parallelogram ACDB. Then is CD = AB or AP,

the height due to the velocity in the curve at A; and CD is also the height due to the perpendicular velocity at D, which must be equal to the former; but by the last corol. the velocity at A is to the perpendicular velocity at D, as AC to 2CD; and as these velocities are equal, therefore AC or BD is equal to 2CD, or 2AB; and hence AB or AP is equal to $\frac{1}{2}$BD, or $\frac{1}{4}$ of the parameter of the diameter AB, by corol. to theor. 13 of the parabola.

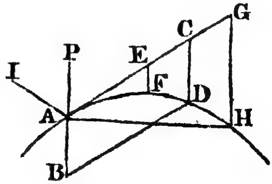
163. *Corol.* 1. Hence, and from cor. 2, theor. 13 of the parabola, it appears that, if from the directrix of the parabola which is the path of the projectile, several lines HE be drawn perpendicular to the directrix, or parallel to the axis; then the velocity of the projectile in the direction of the curve, at any point E, is always equal to the velocity acquired by a body falling freely through the perpendicular line HE.



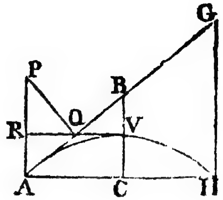
164. *Corol.* 2. If a body, after falling through the height PA (last fig. but one), which is equal to AB, and when it arrives at A, have its course changed, by reflection from an elastic plane AI, or otherwise, into any direction AC, without altering the velocity; and if AC be taken = 2AP or 2AB, and the parallelogram be completed; then the body will describe the parabola passing through the point D.

165. *Corol.* 3. Because AC = 2AB or 2CD or 2AP, therefore $AC^2 = 2AP \times 2CD$ or AP . 4CD; and, because all the perpendiculars EF, CD, GH, are as AE^2, AC^2, AG^2; therefore also AP . 4EF = AE^2, and AP . 4GH = AG^2, &c: and, because the rectangle of the extremes is equal to the rectangle of the means of four proportionals, therefore always

it is AP : AE :: AE : 4EF,
and AP : AC :: AC : 4CD,
and AP : AG :: AG : 4GH,
and so on.



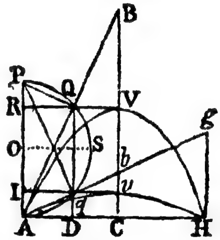
166. Prop. Having given the direction, and the impetus, or altitude due to the first velocity of a projectile; to determine the greatest height to which it will rise, and the random or horizontal range.



Let AP be the height due to the projectile velocity at A, AG the direction, and AH the horizon. On AG let fall the perpendicular PQ, and on AP the perpendicular QR; so shall AR be equal to the greatest altitude CV, and 4QR equal to the horizontal range AH. Or, having drawn PQ perp. to AG, take AG = 4AQ, and draw GH perp. to AH; then AH is the range.

For, by the last corollary, AP : AG : : AG : 4GH;
and, by similar triangles, AP : AG : : AQ : GH,
or - - - AP : AG : : 4AQ : 4GH;
therefore AG = 4AQ; and, by similar triangles, AH = 4QR.

Also, if V be the vertex of the parabola, then AB or $\frac{1}{2}$AG = 2AQ, or AQ = QB; consequently AR = BV, which is = CV by the property of the parabola.



167. *Corol.* 1. Because the angle Q is a right angle, which is the angle in a semicircle, therefore if, on AP as a diameter, a semicircle be described, it will pass through the point Q.

168. *Corol.* 2. If the horizontal range and the projectile velocity be given, the direction of the piece so as to hit the object H, will be thus easily found: Take AD = $\frac{1}{4}$AH, draw DQ perpendicular to AH, meeting the semicircle, described on the diameter AP, in Q and *q*; then AQ or A*q* will be the direction of the piece. And hence it appears, that there are two directions AB, A*b*, which, with the same projectile velocity, give the very same horizontal range AH. And these two directions make equal angles *q*AD, QAP with AH and AP, because the arc PQ = the arc A*q*.

169. *Corol.* 3. Or, if the range AH, and direction AB, be given; to find the altitude and velocity or impetus. Take AD = $\frac{1}{4}$AH, and erect the perpendicular DQ, meeting AB in Q; so shall DQ be equal to the greatest altitude CV. Also, erect AP perpendicular to AH, and QP to AQ; so shall AP be the height due to the velocity.

170. *Corol.* 4. When the body is projected with the same velocity, but in different directions: the horizontal ranges AH will be as the sines of double the angles of elevation.—Or, which is the same, as the rectangle of the sine and co-

sine of elevation. For AD or RQ, which is $\frac{1}{4}$AH, is the sine of the arc AQ, which measures double the angle QAD of elevation.

And when the direction is the same, but the velocities different; the horizontal ranges are as the square of the velocities, or as the height AP, which is as the square of the velocity; for the sine AD or RQ or $\frac{1}{4}$AH is as the radius or as the diameter AP.

Therefore, when both are different, the ranges are in the compound ratio of the squares of the velocities, and the sines of double the angles of elevation.

171. *Corol.* 5. The greatest range is when the angle of elevation is 45°, or half a right angle; for the double of 45 is 90, which has the greatest sine. Or the radius OS, which is $\frac{1}{4}$ of the range, is the greatest sine.

And hence the greatest range, or that at an elevation of 45°, is just double the altitude AP which is due to the velocity, or equal to 4VC. Consequently, in that case, C is the focus of the parabola, and AH its parameter. Also the ranges are equal, at angles equally above and below 45°.

172. *Corol.* 6. When the elevation is 15°, the double of which, or 30°, has its sine equal to half the radius; consequently then its range will be equal to AP, or half the greatest range at the elevation of 45°; that is, the range at 15°, is equal to the impetus or height due to the projectile velocity.

173. *Corol.* 7. The greatest altitude CV, being equal to AR, is as the versed sine of double the angle of elevation, and also as AP or the square of the velocity. Or as the square of the sine of elevation, and the square of the velocity; for the square of the sine is as the versed sine of the double angle.

174. *Corol.* 8. The time of flight of the projectile, which is equal to the time of a body falling freely through GH or 4CV, four times the altitude, is therefore as the square root of the altitude, or as the projectile velocity and sine of the elevation.

SCHOLIUM.

175. From the last proposition and its corollaries may be deduced the following set of theorems, for finding all the circumstances of projectiles on horizontal planes, having any two of them given. Thus, let s, c, t, denote the sine, cosine, and tangent of elevation; S, v the sine and versed sine of the double elevation; R the horizontal range; T the time of

flight; v the projectile velocity; H the greatest height of the projectile; $g = 32\frac{1}{6}$ feet, and a the impetus, or the altitude due to the velocity v. Then,

$$R = 2as = 4asc = \frac{sv^2}{g} = \frac{scv^2}{\frac{1}{2}g} = \frac{\frac{1}{2}gcT^2}{s} = \frac{\frac{1}{2}gT^2}{t} = \frac{4H}{t}.$$

$$v = \sqrt{2ag} = \sqrt{\frac{gR}{s}} = \sqrt{\frac{gR}{2sc}} = \frac{\frac{1}{2}gT}{s} = \frac{2}{s}\sqrt{\tfrac{1}{2}gH}.$$

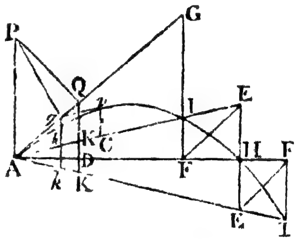
$$T = \frac{{}^{*}sv}{\frac{1}{2}g} = 2s\sqrt{\frac{a}{\frac{1}{2}g}} = \sqrt{\frac{tR}{\frac{1}{2}g}} = \sqrt{\frac{sR}{\frac{1}{2}gc}} = 2\sqrt{\frac{H}{\frac{1}{2}g}}.$$

$$H = as^2 = \tfrac{1}{2}av = \tfrac{1}{4}tR = \frac{sR}{4c} = \frac{s^2v^2}{2g} = \frac{tv^2}{4g} = \tfrac{1}{8}gT^2.$$

And from any of these, the angle of direction may be found. Also, in these theorems, g may, in many cases, be taken $= 32$, without the small fraction $\frac{1}{6}$, which will be near enough for common use.

176. PROP. To determine the range on an oblique plane; having given the impetus or velocity, and the angle of direction.

Let AE be the oblique plane, at a given angle, either above or below the horizontal plane AH; AG the direction of the piece, and AP the altitude due to the projectile velocity at A.



By the last proposition, find the horizontal range AH to the given velocity and direction; draw HE perpendicular to AH, meeting the oblique plane in E; draw EF parallel to AG, and FI parallel to HE; so shall the projectile pass through I, and the range on the oblique plane will be AI. As is evident by theor. 15 of the Parabola, where it is proved, that if AH, AI be any two lines terminated at the curve, and IF, HE parallel to the axis; then is EF parallel to the tangent AG.

177. *Otherwise*, without the Horizontal Range.

Draw PQ perp. to AG, and QD perp. to the horizontal plane AF, meeting the inclined plane in K; take AE = 4AK,

* This time, with 30° elevation, is just equal to the time of perpendicular ascent, with the same velocity v.

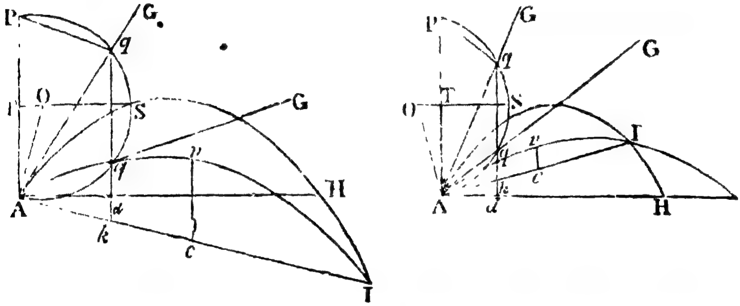
draw EF parallel to AG, and FI parallel to AP or DQ; so shall AI be the range on the oblique plane. For AH = 4AD, therefore EH is parallel to FI, and so on, as above.

Otherwise.

178. Draw Pq making the angle APq = the angle GAI; then take AG = 4Aq, and draw GI perp. to AH. Or, draw qk perp. to AH, and take AI = 4Ak. Also kq will be equal to cv the greatest height above the plane.

For, by cor. 2, art. 164, AP : AG :: AG : 4GI;
and by sim. triangles, AP : AG :: Aq : GI,
or - - - AP : AG :: 4Aq : 4GI;
therefore AG = 4Aq; and by sim. triangles, AI = 4Ak.

Also, qk, or $\frac{1}{4}$GI, is = to cv by theor. 13 of the Parabola.



179. *Corol.* 1. If AO be drawn perp. to the plane AI, and AP be bisected by the perpendicular STO; then with the centre O describing a circle through A and P, the same will also pass through q, because the angle GAI, formed by the tangent AI and AG, is equal to the angle APq, which will therefore stand on the same arc Aq.

180. *Corol.* 2. If there be given the range AI and the velocity, or the impetus, the direction will hence be easily found thus: Take Ak = $\frac{1}{4}$AI, draw kq perp. to AH, meeting the circle described with the radius AO in two points q and q; then Aq or Aq will be the direction of the piece. And hence it appears that there are two directions, which, with the same impetus, give the very same range AI. And these two directions make equal angles with AI and AP, because the arc Pq is equal the arc Aq. They also make equal angles with a line drawn from A through S, because the arc Sq is equal the arc Sq.

181. *Corol.* 3. Or, if there be given the range AI, and the direction Aq; to find the velocity or impetus. Take Ak =

$\frac{1}{4}$AI, and erect kq perp. to AH, meeting the line of direction in q; then draw qP making the $\angle$ AqP $=$ $\angle$ Akq; so shall AP be the impetus, or the altitude due to the projectile velocity.

182. *Corol.* 4. The range on an oblique plane, with a given elevation, is directly proportional to the rectangle of the cosine of the direction of the piece above the horizon, and the sine of the direction above the oblique plane, and reciprocally to the square of the cosine of the angle of the plane above or below the horizon.

For, put $s =$ sin. $\angle q$AI or APq,
$c =$ cos. $\angle q$AH or sin. PAq,
C $=$ cos. $\angle$ IAH or sin. Akd or Akq or AqP.

Then, in the triangle APq, C : s : : AP : Aq;
and in the triangle Akq, C : c : : Aq : Ak;
theref. by composition, $C^2 : cs$: : AP : AK $= \frac{1}{4}$AI.

So that the oblique range AI $= \frac{cs}{C^2} \times 4$AP.

183. The range is the greatest when Ak is the greatest: that is, when kq touches the circle in the middle point S; and then the line of direction passes through S, and bisects the angle formed by the oblique plane and the vertex. Also, the ranges are equal at equal angles above and below this direction for the maximum.

184. *Corol.* 5. The greatest height cv or kq of the projectile, above the plane, is equal to $\frac{s^2}{C^2} \times$ AP. And therefore it is as the impetus and square of the sine of direction above the plane directly, and square of the cosine of the plane's inclination reciprocally.

For - C (sin. AqP) : s (sin. APq) : : AP : Aq,
and C (sin. Akq) : s (sin. kAq) : : Aq : kq,
theref. by comp. $C^2 : s^2$: : AP : kq.

185. *Corol.* 6. The time of flight in the curve AvI is $=$ $\frac{2s}{C}\sqrt{\frac{\text{AP}}{\frac{1}{2}g}}$, where $\frac{1}{2}g = 16\frac{1}{12}$ feet. And therefore it is as the velocity and sine of direction above the plane directly, and cosine of the plane's inclination reciprocally. For the time of describing the curve, is equal to the time of falling freely through GI or $4kq$ or $\frac{4s^2}{C^2} \times$ AP. Therefore, the time being

as the square root of the distance,

$$\sqrt{\tfrac{1}{2}g} : \frac{2s}{\mathrm{C}}\sqrt{\mathrm{AP}} :: 1'' : \frac{2s}{\mathrm{C}}\sqrt{\frac{\mathrm{AP}}{\frac{1}{2}g}},$$ the time of flight.

SCHOLIUM.

186. From the foregoing corollaries may be collected the following set of theorems, relating to projects made on any given inclined planes, either above or below the horizontal plane. In which the letters denote as before, namely,

c = cos. of direction above the horizon,
C = cos. of inclination of the plane,
s = sin. of direction above the plane,
R = the range on the oblique plane,
T the time of flight,
V the projectile velocity,
H the greatest height above the plane,
a the impetus, or alt. due to the velocity V,
$g = 32\tfrac{1}{6}$ feet. Then,

$$\mathrm{R} = \frac{cs}{\mathrm{C}^2} \times 4a = \frac{2cs}{\mathrm{C}^2 g}\mathrm{V}^2 = \frac{gc}{2s}\mathrm{T}^2 = \frac{4c}{s}\mathrm{H}.$$

$$\mathrm{H} = \frac{s^2}{\mathrm{C}^2}a = \frac{s^2\mathrm{V}^2}{2g\mathrm{C}^2} = \frac{s\mathrm{R}}{4c} = \frac{g}{8}\mathrm{T}^2.$$

$$\mathrm{V} = \sqrt{2ag} = \mathrm{C}\sqrt{\frac{g\mathrm{R}}{2cs}} = \frac{g\mathrm{C}}{2s}\mathrm{T} = \frac{2\mathrm{C}}{s}\sqrt{\tfrac{1}{2}g\mathrm{H}}.$$

$$\mathrm{T} = \frac{2s}{\mathrm{C}}\sqrt{\frac{a}{\frac{1}{2}g}} = \frac{s\mathrm{V}}{\frac{1}{2}g\mathrm{C}} = \sqrt{\frac{s\mathrm{R}}{\frac{1}{2}gc}} = 2\sqrt{\frac{\mathrm{H}}{\frac{1}{2}g}}.$$

And from any of these, the angle of direction may be found.

187. Geometrical constructions of the principal cases in projectiles in a non-resisting medium, flow readily from the properties of the parabola; and in many cases those constructions suggest simple modes of computation. The following problems will serve by way of exercise.

1. Given the impetus and elevation; to find, by construction, the range, on a horizontal plane, the greatest height, and thence the time of flight.

2. Given the impetus, and the range, on a horizontal plane; to find, by construction, the elevation, and the greatest height.

3. Given the elevation, and the range on a horizontal plane; to find, by construction, the impetus, the greatest height, and thence by computation, the time.

4. Given the impetus, the point and direction of projection,

to find the place where the ball will fall upon any plane given in position.

5. Given the impetus and the point of projection, to find the elevation necessary to hit any given point; and to show the limits of possibility. Both construction and mode of computation are required.

PRACTICAL GUNNERY.

188. We have now given the whole theory of projectiles, with theorems for all the cases, regularly arranged for use, both for oblique and horizontal planes. But, before they can be applied in resolving the several cases in the practice of gunnery, it is necessary that some more data be laid down, as derived from good experiments made with balls or shells discharged from cannon or mortars, by gunpowder, under different circumstances. For, without such experiments and data, those theorems can be of very little utility in real practice, on account of the imperfections and irregularities in the firing of gunpowder, and the expulsion of balls from guns, but more especially on account of the enormous resistance of the air to all projectiles made with any velocities that are considerable. As to the cases in which projectiles are made with small velocities, or such as do not exceed 200, or 300, or 400 feet per second of time, they may be resolved tolerably near the truth, especially for the larger shells, by the parabolic theory, laid down above. But, in cases of great projectile velocities, that theory is quite inadequate, without the aid of several data drawn from many and good experiments. For so great is the effect of the resistance of the air to projectiles of considerable velocity, that some of those which in the air range only between 2 and 3 miles at the most, would in vacuo range about ten times as far, or between 20 and 30 miles.

The effects of this resistance are also various, according to the velocity, the diameter, and the weight of the projectile. So that the experiments made with one size of ball or shell, will not serve for another size, though the velocity should be the same; neither will the experiments made with one velocity, serve for other velocities, though the ball be the same. And therefore it is plain that, to form proper rules for practical gunnery, we ought to have good experiments made with each size of mortar, and with every variety of charge, from the least to the greatest. And not only so, but these ought also to be repeated at many different angles of elevation, namely, for every single degree between 30° and 60° elevation,

and at intervals of 5° above 60° and below 30°, from the vertical direction to point blank. By such a course of experiments it will be found, that the greatest range, instead of being constantly that at an elevation of 45°, as in the parabolic theory, will be at all intermediate degrees between 45 and 30, being more or less, both according to the velocity and the weight of the projectile; the smaller velocities and larger shells ranging farthest when projected almost at an elevation of 45°; while the greatest velocities, especially with the smaller shells, range farthest with an elevation of about 30°, or little more.

189. There have, at different times, been made certain small parts of such a course of experiments as is hinted at above. Such as the experiments or practice carried on in the year 1773, on Woolwich Common; in which all the sizes of mortars were used, and a variety of small charges of powder. But they were all at the elevation of 45°; consequently these are defective in the higher charges, and in all the other angles of elevation.

Other experiments were also carried on in the same place in the years 1784 and 1786, with various angles of elevation indeed, but with only one size of mortar, and only one charge of powder, and that but a small one too: so that all those nearly agree with the parabolic theory. Other experiments have also been carried on with the ballistic pendulum, at different times; from which have been obtained some of the laws for the quantity of powder, the weight and velocity of the ball, the length of the gun, &c. Namely, that the velocity of the ball varies as the square root of the charge directly, and as the square root of the weight of ball reciprocally; and that, some rounds being fired with a medium length of one-pounder gun, at 15° and 45° elevation, and with 2, 4, 8, and 12 ounces of powder, gave nearly the velocities, ranges, and times of flight, as they are here set down in the following table. But good experiments are wanted with large balls and shells.

Powder.	Elevation of gun.	Velocity of ball.	Range.	Time of flight.
oz.		feet.	feet.	
2	15°	860	4100	9″
4	15	1230	5100	12
8	15	1640	6000	$14\frac{1}{2}$
12	15	1680	6700	$15\frac{1}{2}$
2	45	860	5100	21

190. But as we are not yet provided with a sufficient number and variety of experiments, on which to establish true rules for practical gunnery, independent of the parabolic theory, we must at present content ourselves with the data of some one certain experimented range and time of flight, at a given angle of elevation; and then, by help of these, and the rules in the parabolic theory, determine the like circumstances for other elevations that are not greatly different from the former, assisted by the following practical rules.—

191. SOME PRACTICAL RULES IN GUNNERY.

I. *To find the Velocity of any Shot or Shell.*

RULE. DIVIDE double the weight of the charge of powder by the weight of the shot, both in lbs. Extract the square root of the quotient. Multiply that root by 1600, and the product will be the velocity in feet, or the number of feet the shot passes over per second, nearly.

Or say—As the root of the weight of the shot, is to the root of double the weight of the powder, so is 1600 feet, to the velocity*.

II. *Given the Range at One Elevation; to find the Range at Another Elevation.*

RULE. As the sine of double the first elevation, is to its range; so is the sine of double another elevation, to its range.

III. *Given the Range for one Charge; to find the Range for Another Charge, or the Charge for Another Range.*

RULE. The ranges have the same proportion as the charges; that is, as one range is to its charge, so is any other range to its charge: the elevation of the piece being the same in both cases.

* In more recent experiments carried on at Woolwich, by the Editor of the present edition, in conjunction with the select committee of artillery officers, it has been found that a charge of a third of the weight of the ball, gives, at a medium, a velocity of 1600 feet: gunpowder being much improved in its manufacture since the time when Sir Tho. Blomfield and Dr. Hutton made their experiments. Putting B for the weight of the ball, and C for that of the charge, $v = 1600\sqrt{\frac{3C}{B}}$, is now found a good approximative theorem for the initial velocity.

192. EXAMPLE 1. If a ball of 1lb. acquire a velocity of 1600 feet per second, when fired with 8 ounces of powder; it is required to find with what velocity each of the several kinds of shells will be discharged by the full charges of powder, viz.

Nature of the shells in inches	13	10	8	$5\frac{1}{2}$	$4\frac{2}{5}$
Their weight in lbs. - -	196	90	48	16	8
Charge of powder in lbs. -	9	4	2	1	$\frac{1}{2}$
Ans. The velocities are -	485	477	462	566	566

EXAM. 2. If a shell be found to range 1000 yards when discharged at an elevation of 45°; how far will it range when the elevation is 30° 16′, the charge of powder being the same? Ans. 2612 feet, or 871 yards.

EXAM. 3. The range of a shell, at 45° elevation, being found to be 3750 feet; at what elevation must the piece be set, to strike an object at the distance of 2810 feet, with the same charge of powder? Ans. at 24° 16′, or at 65° 44′.

EXAM. 4. With what impetus, velocity, and charge of powder, must a 13-inch shell be fired, at an elevation of 32° 12′, to strike an object at the distance of 3250 feet?
Ans. impetus 1802, veloc. 340, charge 4lb. $7\frac{1}{2}$oz.

EXAM. 5. A shell being found to range 3500 feet, when discharged at an elevation of 25° 12′; how far then will it range at an elevation of 36° 15′ with the same charge of powder? Ans. 4332 feet.

EXAM. 6. If, with a charge of 9lb. of powder, a shell range 4000 feet; what charge will suffice to throw it 3000 feet, the elevation being 45° in both cases?
Ans. $6\frac{3}{4}$lb. of powder.

EXAM. 7. What will be the time of flight for any given range, at the elevation of 45°, or for the greatest range?
Ans. the time in secs. is $\frac{1}{4}$ the sq. root of the range in feet.

EXAM. 8. In what time will a shell range 3250 feet, at an elevation of 32°? Ans. $11\frac{1}{4}$ sec. nearly.

EXAM. 9. How far will a shot range on a plane which ascends 8° 15′, and another which descends 8° 15′; the impetus being 3000 feet, and the elevation of the piece 32° 30′?
Ans. 4244 feet on the ascent,
and 6745 feet on the descent.

EXAM. 10. How much powder will throw a 13-inch shell

4244 feet on an inclined plane, which ascends 8° 15′, the elevation of the mortar being 32° 30′?

Ans. 7·3765lb. or 7lb. 6oz.

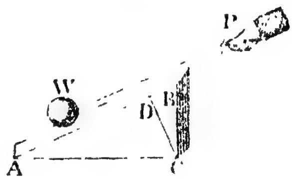
EXAM. 11. At what elevation must a 13-inch mortar be pointed, to range 6745 feet, on a plane which descends 8° 15′; the charge $7\frac{3}{8}$lb. of powder? Ans. 32° 41′$\frac{1}{2}$.

EXAM. 12. In what time will a 13-inch shell strike a plane which rises 8° 30′, when elevated 45°, and discharged with an impetus of 2304 feet? Ans. $14\frac{2}{5}$ seconds.

THE DESCENT OF BODIES ON INCLINED PLANES AND CURVE SURFACES.—THE MOTION OF PENDULUMS.

193. PROP. IF a weight W be sustained on an inclined plane AB, by a power P, acting in a direction WP, parallel to the plane. Then

The weight of the body, W	The length AB,
The sustaining power P, and	The height BC, and
The pressure on the plane, p,	The base AC,
are respectively as	of the plane.



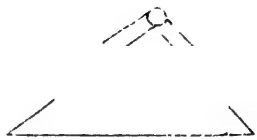
For, draw CD perpendicular to the plane. Now here are three forces, keeping one another in equilibrio; namely, the weight, or force of gravity, acting perpendicular to AC, or parallel to BC; the power acting parallel to DB; and the pressure perpendicular to AB, or parallel to DC: but when three forces keep one another in equilibrio, they are proportional to the sides of the triangle CBD, made by lines in the direction of those forces, by art. 30; therefore those forces are to one another as BC, BD, CD. But the two triangles ABC, CBD, are equiangular, and have their like sides proportional; therefore the three BC, BD, CD, are to one another respectively as the three AB, BC, AC; which therefore are as the three forces W, P, p.

Corol. 1. Hence the weight W, power P, and pressure p, are respectively as radius, sine and cosine, of the plane's elevation BAC above the horizon.

For, since the sides of triangles are as the sines of their opposite angles, therefore the three AB, BC, AC,
are respectively as - - sin. C, sin. A, sin. B,
or as - - - - - radius, sine, consine,
of the angle A of elevation.

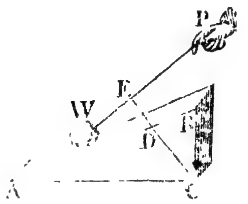
Corol. 2. The power or relative weight that urges a body W down the inclined plane, is $= \frac{BC}{AB} \times W$; or the force with which it descends, or endeavours to descend, is as the sine of the angle A of inclination.

Corol. 3. Hence, if there be two planes of the same height, and two bodies be laid on them which are proportional to the lengths of the planes; they will have an equal tendency to descend down the planes.



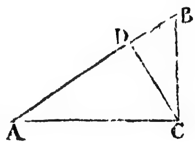
And consequently they will mutually sustain each other if they be connected by a string acting parallel to the planes.

Corol. 4. In like manner, when the power P acts in any other direction whatever, WP; by drawing CDE perpendicular to the direction WP, the three forces in equilibrio, namely, the weight W, the power P, and the pressure on the plane, will still be respectively as AC, CD, AD, drawn perpendicular to the direction of those forces.



194. Prop. The velocity acquired by a body descending freely down an inclined plane AB, is to the velocity acquired by a body falling perpendicularly, in the same time; as the height of the plane BC, is to its length AB.

For the force of gravity, both perpendicularly and on the plane, is constant; and these two, by corol. 2, art. 193, are to each other as AB to BC. But, by art. 131, the velocities generated by any constant forces, in the same time, are as those forces. Therefore the velocity down BA is to the velocity down BC, in the same time, as the force on BA to the force on BC: that is, as BC to BA.



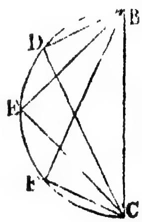
Corol. 1. Hence, as the motion down an inclined plane is produced by a constant force, it will be a motion uniformly accelerated; and therefore the laws before laid down for

accelerated motions in general, hold good for motions on inclined planes; such, for instance, as the following: That the velocities are as the times of descending from rest; that the spaces descended are as the squares of the velocities, or squares of the times; and that if a body be thrown up an inclined plane, with the velocity it acquired in descending, it will lose all its motion, and ascend to the same height, in the same time, and will repass any point of the plane with the same velocity as it passed it in descending.

Corol. 2. Hence also, the space descended along an inclined plane, is to the space descended perpendicularly, in the same time, as the height of the plane CB, to its length AB, or as the sine of inclination to radius. For the spaces described by any forces, in the same time, are as the forces, or as the velocities.

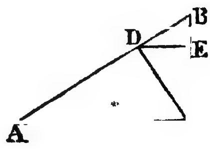
Corol. 3. Consequently the velocities and spaces descended by bodies down different inclined planes, are as the sines of elevation of the planes.

Corol. 4. If CD be drawn perpendicular to AB; then, while a body falls freely through the perpendicular space BC, another body will, in the same time, descend down the part of the plane BD. For by similar triangles, - - - BC : BD :: BA : BC, that is, as the space descended, by corol. 2.



Or, in any right-angled triangle BDC, having its hypothenuse BC perpendicular to the horizon, a body will descend down any of its three sides BD, BC, DC, in the same time. And therefore, if on the diameter BC a circle be described, the time of descending down any chords BD, BE, BF, DC, EC, FC, &c, will be all equal, and each equal to the time of falling freely through the perpendicular diameter BC. Also the velocities acquired in descending down the chords BD, BE, BF, BC, are to one another as the lengths of those chords.

195. PROP. The time of descending down the inclined plane BA, is to the time of falling through the height of the plane BC, as the length BA is to the height BC.



Draw CD perpendicular to AB. Then the times of describing BD and BC are equal, by the last corol. Call that time t, and the time of describing BA call T.

Now, because the spaces described

by constant forces, are as the squares of the times; therefore $t^2 : T^2 :: BD : BA$.

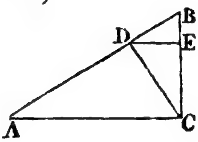
But the three BD, BC, BA, are in continual proportion; therefore $BD : BA :: BC^2 : BA^2$;
hence, by equality, $t^2 : T^2 :: BC^2 : BA^2$,
or - - $t : T :: BC : BA$.

Corol. Hence the times of descending different planes, of the same height, are to one another as the lengths of the planes.

196. Prop. A body acquires the same velocity in descending down any inclined plane BA, as by falling perpendicular through the height of the plane BC.

For, the velocities generated by any constant forces, are in the compound ratio of the forces and times of acting. But if we put

F to denote the whole force of gravity in BC,
f the force on the plane AB,
t the time of describing BC, and
T the time of descending down AB;
then by art. 193, $F : f :: BA : BC$;
and by art. 195, $t : T :: BC : BA$;
theref. by comp. $Ft : fT :: 1 : 1$.



That is, the compound ratio of the forces and times, or the ratio of the velocities, is a ratio of equality.

Corol. 1. Hence the velocities acquired, by bodies descending down any planes, from the same height, to the same horizontal line, are equal.

Corol. 2. If the velocities be equal, at any two equal altitudes, D, E; they will be equal at all other equal altitudes A, C.

Corol. 3. Hence also, the velocities acquired by descending down any planes, are as the square roots of the heights.

SCHOLIUM.

197. We may here introduce some useful formulæ, relative to motions along inclined planes, analogous to those already given for bodies falling freely (art. 154, 158.)

I. Let g, as before $= 32\frac{1}{6}$ feet, s the space along an inclined plane whose inclination is i, t the time, v the velocity; then

$$1.\ s = \tfrac{1}{2}gt^2 \sin. i = \frac{v^2}{2g \sin. i} = \tfrac{1}{2}tv$$

2. $v = gt \sin. i = \sqrt{(2gs \sin. i)} = \frac{2s}{t}$

3. $t = \sqrt{\frac{2s}{g \sin. i}} = \frac{2s}{v}$.

II. Suppose V to be the velocity with which a body is projected up or down the plane; then, we have

4. $v = \text{V} \mp gt \sin. i$

5. $s = \text{V}t \mp \frac{1}{2}gt \sin. i = \frac{\text{V}^2 \mp v^2}{2g \sin. i}$.

Making $v = 0$, in equa. 4, and the latter member of equa. 5; the first will give the *time* at which the body will cease to rise, the latter the *space*.

III. If R be a constant resistance to motion on a horizontal plane, then

6. $v = \text{V} - \text{R}t$

7. $s = \text{V}t - \frac{1}{2}\text{R}t^2 = \frac{\text{V}^2 - v^2}{2\text{R}}$,

where, making $v = 0$, we find when the motion ceases.

198. The first eight of the following problems will serve to exemplify these theorems.

1. How far will a body descend from quiescence in 4 seconds, along an inclined plane whose length is 400 and height 300 feet?

2. What velocity will such a body have acquired when it has reached the bottom A of the plane? (fig. to art. 194.)

3. Suppose AD = DB, in what time will the body pass over each of those portions?

4. How long would a body be in falling down 100 feet of a plane whose length AB is 150 feet, and height BC 60?

5. If AB = 90, and BC = 25 feet, what velocity would a body acquire in falling through 70 feet?

6. A body is projected up an inclined plane, whose length is 10 times its height, with a velocity of 30 feet per second; in what time will its velocity be destroyed, and it cease to ascend?

7. Suppose that at the moment a body is projected up AB with the velocity acquired by falling down it, another body begins to fall down it, where will they meet, the length of AB being given?

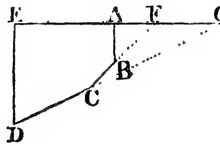
8. Given AB = 90, BC = 60 feet. And suppose two bo-

dies to be let fall the same moment, one vertically, the other down the plane CA; what distance CD will the latter have moved, when the former has descended to B?

9. Ascertain, geometrically, the position of the right line of quickest descent, from a given point to a given plane.

10. Find, geometrically, the slope of a roof, down which rain may descend quickest.

199. PROP. If a body descend down any number of contiguous planes, AB, BC, CD; it will at last acquire the same velocity, as a body falling perpendicularly through the same height ED, supposing the velocity not altered by changing from one plane to another.



Produce the planes DC, CB, to meet the horizontal line EA produced in F and G. Then, by cor. 1, last art. the velocity at B is the same, whether the body descend through AB or FB. And therefore the velocity at C will be the same, whether the body descend through ABC or through FC, which is also again the same as by descending through GC. Consequently it will have the same velocity at D, by descending through the planes AB, BC, CD, as by descending through the plane GD; supposing no obstruction to the motion by the body impinging on the planes at B and C: and this again, is the same velocity as by descending through the same perpendicular height ED.

Corol. 1. If the lines ABCD, &c, be supposed indefinitely small, they will form a curve line, which will be the path of the body; from which it appears that a body acquires also the same velocity in descending along any curve, as in falling perpendicularly through the same height.

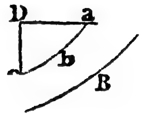
Corol. 2. Hence also, bodies acquire the same velocity by descending from the same height, whether they descend perpendicularly, or down any planes, or down any curve or curves. And if their velocities be equal, at any one height, they will be equal at all other equal heights. Therefore the velocity acquired by descending down any lines or curves, are as the square roots of the perpendicular heights.

Corol. 3. And a body, after its descent through any curve, will acquire a velocity which will carry it to the same height through an equal curve, or through any other curve, either by running up the smooth concave side, or by being retained in the curve by a string, and vibrating like a pendulum:

Also, the velocities will be equal, at all equal altitudes; and the ascent and descent will be performed in the same time, if the curves be the same.

200. Prop. The times in which bodies descend through similar parts of similar curves, ABC, *abc*, placed alike, are as the square roots of their lengths.

That is, the time in AC is to the time in *ac*, as $\sqrt{AC}$ to $\sqrt{ac}$.



For, as the curves are similar, they may be considered as made up of an equal number of corresponding parts, which are every where, each to each, proportional to the whole. And as they are placed alike, the corresponding small similar parts will also be parallel to each other. But the time of describing each of these pairs of corresponding parallel parts, by art. 194, *cor.* 1, are as the square roots of their lengths, which, by the supposition, are as $\sqrt{AC}$ to $\sqrt{ac}$, the roots of the whole curves. Therefore, the whole times are in the same ratio of $\sqrt{AC}$ to $\sqrt{ac}$.

Corol. 1. Because the axes DC, *Dc*, of similar curves, are as the lengths of the similar parts AC, *ac*; therefore the times of descent in the curves AC, *ac*, are as $\sqrt{DC}$ to $\sqrt{Dc}$, or the square roots of their axes.

Corol. 2. As it is the same thing, whether the bodies run down the smooth concave side of the curves, or be made to describe those curves by vibrating like a pendulum, the lengths being DC, D*c*; therefore the times of the vibration of pendulums, in similar arcs of any curves, are as the square roots of the lengths of the pendulums.

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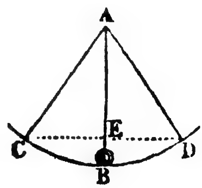
201. Having, in the last corollary, mentioned the pendulum, it may not be improper here to add some remarks concerning it.

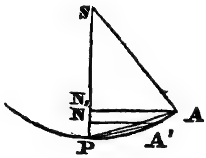


A simple pendulum consists of a small ball, or other heavy body B, hung by a fine string or thread, moveable about a centre A, and describing the arc CBD; by which vibration the same motions happen to this heavy body, as would happen to any body descending by its gravity along the spherical superficies CBD, if that superficies were perfectly hard and smooth. If the

pendulum be carried to the situation AC, and then let fall, the ball in descending will describe the arc CB; and in the point B it will have that velocity which is acquired by descending through CB, or by a body falling freely through EB. This velocity will be sufficient to cause the ball to ascend through an equal arc BD, to the same height D from whence it fell at C; having there lost all its motion, it will again begin to descend by its own gravity; and in the lowest point B it will acquire the same velocity as before; which will cause it to re-ascend to C: and thus, by ascending and descending, it will perform continual vibrations in the circumference CBD. And if the motions of pendulums met with no resistance from the air, and if there were no friction at the centre of motion A, the vibrations of pendulums would never cease. But from these obstructions, though small, it happens, that the velocity of the ball in the point B is a little diminished in every vibration; and consequently it does not return precisely to the same points C or D, but the arcs described continually become shorter and shorter, till at length they are insensible; unless the motion be assisted by a mechanical contrivance, as in clocks, called a maintaining power.

Our present investigations relate to the simple pendulum, above described: the consideration of compound pendulums requires the previous knowledge of the centre of oscillation.

202. PROP. When a pendulum vibrates in a circular arc, the velocities acquired in the lowest point, are as the chords of the semi-arcs described.

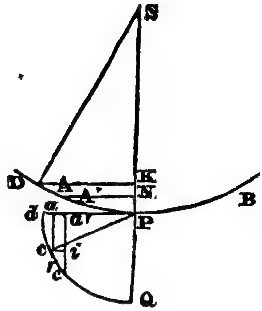


For, the velocity at P of a body that has descended through any arc AP, is equal to the velocity at P of a body that has fallen freely through the versed-sine NP (art. 199, cor. 2.)

Hence, velocity at P after descent through arc AP, is to velocity at P after descent through arc A′P, as $\sqrt{NP}$ to $\sqrt{N'P}$, that is (Geom. th. 87) as chord AP to chord A′P.

203. *Corol.* If, therefore, we would impart to a body a given velocity, v, we have only to compute the height NP, such that $NP = \frac{v^2}{2g} = \frac{v^2}{64\frac{1}{3}\text{ feet}}$, and through the point N draw the horizontal line NA; then AA′P an arc (of any circle passing through P) is one, through which when a body has fallen it will have acquired the proposed velocity. This is extremely useful in experiments on collision.

3. **Prop.** To investigate the time of vibration of a pendulum of given length, in an indefinitely small arc.



Now, in estimating the time of an oscillation in an indefinitely small circular arc, let it be recollected that the excess of such an arc above its chord, being incomparably less than itself, may be neglected; so that we may consider the square of such an arc (like that of its chord, Geom. th. 87) as equal to the rectangle under the versed-sine and the diameter.

Indeed, if instead of indefinitely small arcs we took arcs of 40′ or 50′, and compared the respective differences of their squares and those of their chords, we should find that the error would not exceed the 29000dth part of either result.

Thus, $\text{arc}^2\ 50' - \text{arc}^2\ 40' = 145444^2 - 116355^2$
$= 261799 \times 29089$,
while $\text{chord}^2\ 50' - \text{chord}^2\ 40' = 145442^2 - 116354^2$
$= 261796 \times 29088$.

Let, then, DPB represent such a very short oscillation of a pendulum whose length, l, is SP, S being the point of suspension.

$$\text{Then, versin. KP} = \text{arc}^2\ \text{DP} \div 2l$$
$$\text{versin. NP} = \text{arc}^2\ \text{AP} \div 2l.$$

Their diff. $\text{KN} = \dfrac{\text{DP}^2 - \text{AP}^2}{2l}$; which is the altitude through which a body must fall to acquire the velocity at A. Putting this value of the altitude in the usual expression for falling bodies, $v = \sqrt{(2gs)}$, it becomes $v = \sqrt{\left(2g \cdot \dfrac{\text{DP}^2 - \text{AP}^2}{2l}\right)}$ $= \sqrt{\dfrac{g}{l}} \cdot \sqrt{(\text{DP}^2 - \text{AP}^2)}$. This will be the velocity with which the pendulum will describe an exceedingly minute portion of the arc, such as AA′.

Draw, horizontally, dP = arc DP; with dP as radius describe the quadrantal arc dcc'Q; make da = DA, aa' = AA′, and draw ac, $a'c'$, parallel to PQ.

Then, vel. at A

$$= \sqrt{\frac{g}{l}} \cdot \sqrt{(\text{DP}^2 - \text{AP}^2)} = \sqrt{\frac{g}{l}} \cdot \sqrt{(d\text{P}^2 - a\text{P}^2)} = ac\sqrt{\frac{g}{l}}.$$

But, since time of describing a space as AA$' = aa'$, is inversely as the velocity, or $t = \frac{s}{v}$, we have

$$\text{time through AA}' \text{ (or } aa') = \frac{aa'}{ac}\sqrt{\frac{l}{g}} = \frac{cc'}{c\text{P}}\sqrt{\frac{l}{g}},$$

$$\text{(because, by sim. tri. } \frac{aa'}{ac} = \frac{cc'}{c\text{P}}).$$

The same reasoning applies for every minute successive portion, such as A·A$'$, of the semi-arc described by the pendulum: and when the ball has descended from D to P, the corresponding arc to dP its equal is the quadrant dcc'Q: the expression for the time, therefore, becomes, in that case,

$$t' = \frac{dc\text{Q}}{\text{PQ}}\sqrt{\frac{l}{g}} = \frac{\text{semicircum.}}{\text{diam.}}\sqrt{\frac{l}{g}} = \tfrac{1}{2}\pi\sqrt{\frac{l}{g}}.$$

The time of ascending through PB = PD is, manifestly, equal to the above: therefore, ultimately, the time of complete oscillation through DPB, is,

$$t = \pi\sqrt{\frac{l}{g}}. \quad . \quad . \quad . \quad . \quad (1).$$

Consequently, *the times of oscillation are as the square roots of the lengths of the pendulums,* the force of gravity remaining the same.

204. For the same reason that we have the above equa. when l is the length of the pendulum, and g the lineal measure of the force of gravity, we have $t' = \pi\sqrt{\frac{l'}{g'}}$, in any other place where g' measures the force of gravity, and l' is the length of the pendulum.

Consequently, in general,

$$t : t' :: \sqrt{\frac{l}{g}} : \sqrt{\frac{l'}{g'}}. \quad . \quad . \quad . \quad . \quad (2).$$

If the force of gravity be the same, we have

$$t : t' :: \sqrt{l} : \sqrt{l'}. \quad . \quad . \quad . \quad . \quad (3).$$

If the same pendulum be actuated by different gravitating forces, we have

$$t : t' :: \sqrt{\frac{1}{g}} : \sqrt{\frac{1}{g'}} :: \sqrt{g'} : \sqrt{g}. \quad . \quad . \quad . \quad . \quad (4).$$

When pendulums oscillate in equal times in different places, we have,

$$g : g' :: l : l'. \quad . \quad . \quad . \quad . \quad (5).$$

Other theorems may readily be deduced.

205. If either g or l be determined by experiment, the equa. 1. for t will give the other. Thus, if $\frac{1}{2}g$, or the space fallen through by a heavy body in 1″ of time, be found, then this theorem will give the length of the seconds pendulum. Or, if the length of the seconds pendulum be observed by experiment, which is the easier way; this theorem will give g. Now, in the latitude of London, the length of a pendulum which vibrates seconds, has been found to be $39\frac{1}{8}$ inches; and this being written for l in the theorem, it gives $\pi\sqrt{\dfrac{39\frac{1}{8}}{g}} = 1''$: and hence is found $\frac{1}{2}g = \frac{1}{2}\pi^2 l = \frac{1}{2}\pi^2 \times 39\frac{1}{8} = 193{\cdot}07$ inches $= 16\frac{1}{12}$ feet, for the descent of gravity in 1″; which it has also been found to be very exactly, by many accurate experiments.

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206. Hence is found the length of a pendulum that shall make any number of vibrations in a given time. Or, the number of vibrations that shall be made by a pendulum of a given length. Thus, suppose it were required to find the length of a half seconds pendulum, or a quarter-seconds pendulum; that is, a pendulum to vibrate twice in a second, or 4 times in a second. Then, since the time of vibration is as the square root of the length,

therefore $1 : \frac{1}{2} :: \sqrt{39\frac{1}{8}} : \sqrt{l}$,

or $1 : \frac{1}{4} :: 39\frac{1}{8} : \dfrac{39\frac{1}{8}}{4} = 9\frac{3}{4}$ inches nearly, the length of the half-seconds pendulum.

And $1 : \frac{1}{16} :: 39\frac{1}{8} : 2\frac{7}{16}$ inches, the length of the quarter-seconds pendulum.

Again, if it were required to find how many vibrations a pendulum of 80 inches long will make in a minute. Here

$\sqrt{80} : \sqrt{39\frac{1}{8}} :: 60''$ or $1' : 60\sqrt{\dfrac{39\frac{1}{8}}{80}} = 7\frac{1}{2}\sqrt{31{\cdot}3} =$ - - $41{\cdot}95987$, or almost 42 vibrations in a minute.

207. For military men it is a good practice to have a portable pendulum, made of painted tape with a brass bob at the end, so that the whole except the bob, may be rolled up within a box, and the whole enclosed in a shagreen case. The tape is marked 200, 190, 180, 170, 160, &c. 80, 75, 70, 65, 60, at points, which being assumed respectively as points of suspension, the pendulum will make 200,190, &c. down to 60 vibrations in a minute. Such a portable pendulum is

highly useful in experiments relative to falling bodies, the velocity of sound, &c.

For the comparison of the times of oscillation in indefinitely small arcs of circles, in finite arcs of circles, and in cycloidal arcs, the student may turn to probs. 13 and 14, at the end of this volume, and prob. 42 at the end of vol. III.

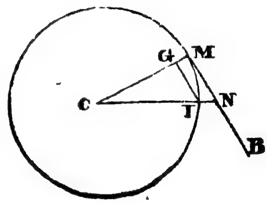
CENTRAL FORCES.

208. *Def.* 1. *Centripetal force* is a force which tends constantly to solicit or to impel a body towards a certain fixed point or centre.

2. *Centrifugal force* is that by which it would recede from such a centre, were it not prevented by the centripetal force.

3. These two forces are, jointly, called *central* forces.

209. PROP. If a body, M, drawn continually towards a fixed point, C, by a constant force, φ, and projected in a direction, MB, perpendicular to CM, describe the circumference of a circle about the centre C, the central force φ, is to the weight of the body, as the altitude due to the velocity of projection, is to half the radius CM.



Let v be the velocity of projection in the tangent MB, and r the radius CM. Independently of the action of the central force, the body would describe, along MB, during the very small time t, a space $\text{MN} = tv$, and would recede from the point C by the quantity IN, which may, without error, be regarded as equal to GM, when the arc MI is exceedingly small. If, therefore, the body instead of moving in the tangent, were kept in the circumference by the central force φ, its operation in the time t, would (art. 130) be equal to $\frac{1}{2}\varphi t^2$, and at the same time $=$ MG. But by the nature of the circle $\text{MG} = \frac{\text{MI}^2}{2r} = \frac{\text{MN}^2}{2r}$ (in an extremely small arc) $= \frac{t^2v^2}{2r}$, by the above.

Making, therefore, $\frac{1}{2}\varphi t^2 = \frac{t^2v^2}{2r}$, it reduces to

$$\varphi = \frac{v^2}{r} \quad . \quad . \quad . \quad . \quad (1).$$

Putting a for the altitude due to the velocity v, since (by art. 154) $v^2 = 2ag$, we have $\varphi = \frac{2ag}{r}$; whence there results

$$\varphi : g :: a : \tfrac{1}{2}r.$$

Thus far, we have, in reality, considered only the unit of mass; but, if we multiply the first two terms of the above proportion by the mass of the body, the whole will still remain a correct proportion, and the general result may be thus enunciated: viz.

The centripetal force of any body, if it be free, or its centrifugal force, if it be retained to the centre C, by a thread (or otherwise), is to the weight of that body, as the height due to the velocity v, is to the half of the radius CM.

210. Hence, it appears that, so long as φ and r remain constant, the velocity v will be constant.

211. If both members of the equation 1 be multiplied by the mass M of the body, and we put F to represent the centrifugal force of that mass, we shall have $F = \frac{Mv^2}{r}$. In like manner, if F' is the centrifugal force of another body which revolves with the velocity v' in a circle whose radius is r', we shall have

$$F : F' :: \frac{v^2}{r} : \frac{v'^2}{r'} \quad . \quad . \quad . \quad . \quad . \quad (2).$$

212. If T and T′ denote the times of revolution of the two bodies, because $v = \frac{2\pi r}{T}$, and $v' = \frac{2\pi r'}{T'}$, we have

$$F : F' :: \frac{r}{T^2} : \frac{r'}{T'^2} \quad . \quad . \quad . \quad . \quad . \quad (3).$$

213. If the times of revolution are equal, we shall have

$$F : F' :: r : r' \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

214. And, if we assume $T^2 : T'^2 :: r^3 : r'^3$, as in the planetary motions, the proportion (3) will become

$$F : F' :: r'^2 : r^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

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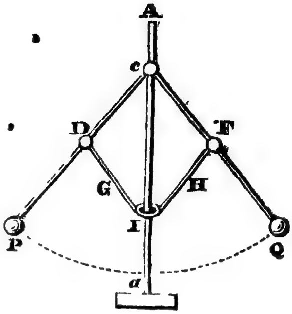
215. The subject of central forces is too extensive and momentous to be adequately pursued here. The student may consult the treatises of mechanics by *Gregory* and *Poisson*, and those on fluxions by *Simpson*, *Dealtry*, &c.

We shall simply present in this place, one example connected with practical mechanics.

Exam. Investigate the characteristic property of a conical pendulum applied as a regulator or *governor* to steam-engines, &c.



This contrivance will be readily comprehended from the marginal figure, where Aa is a vertical shaft capable of turning freely upon the sole a. cD, cF, are two bars which move freely upon the centre c, and carry at their lower extremities two equal weights, P, Q: the bars cD, cF, are united, by a proper articulation, to the bars G, H, which latter are attached to a ring, I, capable of sliding up and down the vertical shaft, Aa. When this shaft and connected apparatus are made to revolve, in virtue of the centrifugal force the balls P, Q, fly out more and more from Aa, as the rotatory velocity increases: if, on the contrary, the rotatory velocity slackens, the balls descend and approach Aa. The ring I *ascends* in the former case, *descends* in the latter: and a lever connected with I may be made to correct appropriately, the energy of the moving power. Thus, in the steam-engine, the ring may be made to act on the valve by which the steam is admitted into the cylinder; to augment its opening when the motion is slackening, and reciprocally diminish it when the motion is accelerated.

The construction is, often, so modified that the flying out of the balls causes the ring I to be depressed, and *vice versa*; but the general principle is the same. If FQ = FI = DP = DI, then I, P, Q, are always in some one horizontal plane: but that is not essential to the construction.

Now, let t denote the time of one revolution of the shaft, x the variable horizontal distance of each ball from that shaft, π as usual $= 3{\cdot}141593$: then will the velocity of each ball be $= \frac{2\pi x}{t}$, and (art. 209.) its centrifugal force $= \left(\frac{2\pi x}{t}\right)^2 \div x = 4\pi^2 x$ The balls being operated upon simultaneously by the centrifugal force and the force of gravity, of which one operates horizontally, the other vertically, the

resultant of the two forces is, evidently, always in the actual position of the handle CD, CF. It follows, therefore, that the ratio of the gravity to the centrifugal force, is that of cos. ICQ to sin. ICQ, or that of the vertical distance of Q below C to its horizontal distance from Aa. Call the former d, the latter being x:

$$\text{then } d : x :: g : \frac{4\pi^2 x}{t^2},$$

$$\text{theref. } \frac{gt^2}{4\pi^2 x} = \frac{d}{x} \text{ and } t = 2\pi\sqrt{\frac{d}{g}} = 1{\cdot}10784\sqrt{d}.$$

Hence, the periodic time varies as the square root of the altitude of the conic pendulum, let the radius of the base be what it may.

Hence, also, when ICQ = ICP = 45°, the centrifugal force of each ball is equal to its weight.

ON THE CENTRES OF PERCUSSION, OSCILLATION, AND GYRATION.

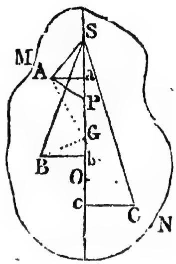
216. *The Centre of Percussion* of a body, or a system of bodies, revolving about a point, or axis, is that point, which striking an immoveable object, the whole mass shall not incline to either side, but rest, as it were, in equilibrio, without acting on the centre of suspension,

217. *The Centre of Oscillation* is that point, in a body vibrating by its gravity, in which if any body be placed, or if the whole mass be collected, it will perform its vibrations in the same time, and with the same angular velocity, as the whole body, about the same point or axis of suspension.

218. *The Centre of Gyration* is that point, in which if the whole mass be collected, the same angular velocity will be generated in the same time, by a given force acting at any place, as in the body or system itself.

219. The *angular motion* of a body, or system of bodies, is the motion of a line connecting any point and the centre or axis of motion; and is the same in all parts of the same revolving body. And in different unconnected bodies, each revolving about a centre, the *angular velocity* is as the absolute velocity directly, and as the distance from the centre inversely; so that, if their absolute velocities be as their radii or distances, the angular velocities will be equal.

220. PROP. To find the centre of percussion of a body or system of bodies.



Let the body revolve about an axis passing through any point S in the line SGO, passing through the centres of gravity and percussion, G and O. Let MN be the section of the body, or the plane in which the axis SGO moves. And conceive all the particles of the body to be reduced to this plane, by perpendiculars let fall from them to the plane: a supposition which will not affect the centres G, O, nor the angular motion of the body.

Let A be the place of one of the particles, so reduced; join SA, and draw AP perpendicular to AS, and Aa perpendicular to SGO: then AP will be the direction of A's motion as it revolves about S; and the whole mass being stopped at O, the body A will urge the point P, forward, with a force proportional to its quantity of matter and velocity, or to its matter and distance from the point of suspension S; that is, as A . SA; and the efficacy of this force in a direction perpendicular to SO, at the point P, is as A . Sa, by similar triangles; also, the effect of this force on the lever, to turn it about O, being as the length of the lever, is as A . Sa . PO = A . Sa . (SO − SP) = A . Sa . SO − A . Sa . SP = A . Sa . SO − A . SA^2. In like manner, the forces of B and C, to turn the system about O, are as

$$\text{B} . \text{S}b . \text{SO} - \text{B} . \text{SB}^2, \text{ and}$$
$$\text{C} . \text{S}c . \text{SO} - \text{C} . \text{SC}^2, \&c.$$

But, since the forces on the contrary sides of O destroy one another, by the definition of this force, the sum of the positive parts of these quantities must be equal to the sum of the negative parts,

that is, $\text{A} . \text{S}a . \text{SO} + \text{B} . \text{S}b . \text{SO} + \text{C} . \text{S}c . \text{SO} \;\&c = $

$$\text{A} . \text{SA}^2 + \text{B} . \text{SB}^2 + \text{C} . \text{SC}^2 \;\&c; \text{ and}$$

$$\text{hence SO} = \frac{\text{A} . \text{SA}^2 + \text{B} . \text{SB}^2 + \text{C} . \text{SC}^2 \;\&c}{\text{A} . \text{S}a + \text{B} . \text{S}b + \text{C} . \text{S}c \;\&c},$$

which is the distance of the centre of percussion below the axis of motion.

And here it must be observed that, if any of the points A, b, &c, fall on the contrary side of S, the corresponding product A . Sa, or B . Sb, &c, must be made negative.

221. *Corol.* 1. Since, by art. 105, A + B + C &c, or the body b × the distance of the centre of gravity, SG, is = A .

$sa + B \cdot sb + C \cdot sc$ &c, which is the denominator of the value of so; therefore the distance of the centre of percussion, is $so = \frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2 \text{ \&c}}{SG \times \text{body } b}$.

222. *Corol.* 2. Since, by Geometry, theor. 36, 37,

$$\begin{aligned} \text{it is } SA^2 &= SG^2 + GA^2 - 2SG \cdot Ga, \\ \text{and } SB^2 &= SG^2 + GB^2 + 2SG \cdot Gb, \\ \text{and } SC^2 &= SG^2 + GC^2 + 2SG \cdot GC, \text{ \&c}; \end{aligned}$$

and, by cor. 5, art. 101, the sum of the last terms is nothing, namely, $-2SG \cdot Ga + 2SG \cdot Gb + 2SG \cdot GC \text{ \&c} = 0$; therefore the sum of the others, or $A \cdot SA^2 + B \cdot SB^2$ &c

is $= (A + B \text{ \&c}) \cdot SG^2 + A \cdot GA^2 + B \cdot GB^2 + C \cdot GC^2$ &c,

or $= b \cdot SG^2 + A \cdot GA^2 + B \cdot GB^2 + C \cdot GC^2$ &c;

which being substituted in the numerator of the foregoing value of so, gives

$$SO = \frac{b \cdot SG^2 + A \cdot GA^2 + B \cdot GB^2 + \text{\&c}}{b \cdot SG},$$

$$\text{or } SO = SG + \frac{A \cdot GA^2 + B \cdot GB^2 + C \cdot GC^2 \text{ \&c}}{SG}$$

223. *Corol.* 3. Hence the distance of the centre of percussion always exceeds the distance of the centre of gravity, and the excess is always $GO = \frac{A \cdot GA^2 + B \cdot GB^2 \text{ \&c}}{SG}$

224. And hence also, $SG \cdot GO = \frac{A \cdot GA^2 + B \cdot GB^2 \text{ \&c}}{\text{the body } b}$;

that is $SG \cdot GO$ is always the same constant quantity, wherever the point of suspension S is placed; since the point G and the bodies A, B, &c, are constant. Or GO is always reciprocally as SG, that is GO is less, as SG is greater; and consequently the point O rises upwards and approaches towards the point G, as the point S is removed to the greater distance; and they coincide when SG is infinite. But when S coincides with G, then GO is infinite, or O is at an infinite distance.

225. **Prop.** If a body A, at the distance SA from an axis passing through S, perpendicular to the plane of the paper, be made to revolve about that axis by any force acting at P in the line SP, perpendicular to the axis of motion: it is required to determine the quantity or matter of another body, Q, which being placed at P, the point where the force acts, it shall be accelerated in the same manner, as when A revolved at the distance SA; and consequently, that the angular velocity of A and Q about S, may be the same in both cases.

By the nature of the lever, SA : SP : : f : $\frac{SP}{SA} . f$, the effect of the force f, acting at P, on the body at A; that is, the force f acting at P, will have the same effect on the body A, as the force $\frac{SP}{SA} f$, acting directly at the point A. But as ASP revolves altogether about the axis at S, the absolute velocities of the points A and S, or of the bodies A and Q, will be as the radii SA, SP, of the circle described by them. Here then we have two bodies A and Q, which being urged directly by the forces f and $\frac{SP}{SA} f$, acquire velocities which are as SP and SA. And since the motive forces of bodies are as their mass and velocity: therefore - - - -

$\frac{SP}{SA} f : f :: A . SA : Q . SP$, and $SP^2 : SA^2 :: A : Q = \frac{SA^2}{SP^2} A$,

which therefore expresses the mass of matter which, being placed at P, would receive the same angular motion from the action of any force at P, as the body A receives. So that the resistance of any body A, to a force acting at any point P, is directly as the square of its distance SA from the axis of motion, and reciprocally as the square of the distance SP of the point where the force acts.

226. *Corol.* 1. Hence the force which accelerates the point P, is to the force of gravity, as $\frac{f . SP^2}{A . SA^2}$ to 1, or as $f . SP^2$ to $A . SA^2$.

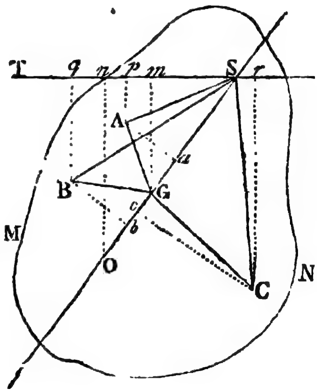
227. *Corol.* 2. If any number of bodies A, B, C, be put in motion, about a fixed axis passing through S, by a force acting at P; the point P will be accelerated in the same manner, and consequently the whole system will have the same angular velocity, if instead of the bodies A, B, C, placed at the distances SA, SB, SC, there be substituted the bodies $\frac{SA^2}{SP^2} A$, $\frac{SB^2}{SP^2} B$, $\frac{SC^2}{SP^2} C$; these being collected into the point P. And hence, the moving force being f, and the matter moved being $\frac{A . SA^2 + B . SB^2 + C . SC^2}{SP^2}$; therefore

$\dfrac{f \cdot \text{SP}^2}{\text{A} \cdot \text{SA}^2 + \text{B} \cdot \text{SB}^2 + \text{C} \cdot \text{SC}^2}$ is the accelerating force; which therefore is to the accelerating force of gravity, as $f \cdot \text{SP}^2$ to $\text{A} \cdot \text{SA}^2 + \text{B} \cdot \text{SB}^2 + \text{C} \cdot \text{SC}^2$.

228. *Corol.* 3. The angular velocity of the whole system of bodies, is as $\dfrac{f \cdot \text{SP}}{\text{A} \cdot \text{SA}^2 + \text{B} \cdot \text{SB}^2 + \text{C} \cdot \text{SC}^2}$. For the absolute velocity of the point P, is as the accelerating force, or directly as the motive force f, and inversely as the mass $\dfrac{\text{A} \cdot \text{SA}^2 \;\&c}{\text{SP}^2}$: but the angular velocity is as the absolute velocity directly, and the radius SP inversely; therefore the angular velocity of P, or of the whole system, which is the same thing, is as $\dfrac{f \cdot \text{SP}}{\text{A} \cdot \text{SA}^2 + \text{B} \cdot \text{SB}^2 + \text{C} \cdot \text{SC}^2}$.

229. PROP. To determine the centre of oscillation of any compound mass, or body MN, or of any system of bodies A, B, C, &c.

Let MN be the plane of vibration, to which let all the matter be reduced, by letting fall perpendiculars from every particle, to this plane. Let G be the centre of gravity, and O the centre of oscillation; through the axis S draw SGO, and the horizontal line Sq; then from every particle A, B, C, &c, let fall perpendiculars Aa, Ap, Bb, Bq, Cc, Cr, to these two lines; and join SA, SB, SC; also, draw Gm, On, perpendicular to Sq. Now the forces of the weights A, B, C, to turn the body about the axis, are A . Sp, B . Sq, — C . Sr; therefore, by cor. 3, art. 228, the angular motion generated by all these forces is $\dfrac{\text{A} \cdot \text{S}p + \text{B} \cdot \text{S}q - \text{C} \cdot \text{S}r}{\text{A} \cdot \text{SA}^2 + \text{B} \cdot \text{SB}^2 + \text{C} \cdot \text{SC}^2}$.



Also, the angular veloc. any particle p, placed in O, generates in the system, by its weight, is $\dfrac{p \cdot \text{S}n}{p \cdot \text{SO}^2}$ or $\dfrac{\text{S}n}{\text{SO}^2}$, or $\dfrac{\text{S}m}{\text{SG} \cdot \text{SO}}$, because of the similar triangles SGm, SOn. But, by the pro-

blem, the vibrations are performed alike in both cases, and therefore these two expressions must be equal to each other, that is, $\frac{sm}{SG \cdot SO} = \frac{A \cdot sp + B \cdot sq - C \cdot sr}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$; and hence

$$SO = \frac{sm}{SG} \times \frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{A \cdot sp + B \cdot sq - C \cdot sr}.$$

But, by cor. 2, art. 105, the sum $A \cdot sp + B \cdot sq - C \cdot sr = (A + B + C) \cdot sm$; therefore the distance SO = - - - -

$$\frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{SG \cdot (A + B + C)} = \frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{A \cdot sa + B \cdot sb + C \cdot sc}$$

by art. 107, which is the distance of the centre of oscillation O, below the axis of suspension; where any of the products $A \cdot sa$, $B \cdot sb$, must be negative, when a, b, &c, lie on the other side of S. So that this is the same expression as that for the distance of the centre of percussion, found in art. 220.

Hence it appears, that the centres of percussion and of oscillation, are in the very same point. And therefore the properties in all the corollaries there found for the former, are to be here understood of the latter; and it will be necessary to mark them carefully, as they are of great practical utility.

230. *Corol.* 1. If p be any particle of a body b, and d its distance from the axis of motion S; also G, O the centres of gravity and oscillation. Then the distance of the centre of oscillation of the body, from the axis of motion, is - - -

$$SO = \frac{\text{sum of all the } pd^2}{SG \times \text{the body } b}.$$

231. *Corol.* 2. If b denote the matter in any compound body, whose centres of gravity and oscillation are G and O; the body P, which being placed at P, where the force acts as in the last proposition, and which receives the same motion from that force as the compound body b, is $P = \frac{SG \cdot SO}{SP^2} \cdot b$.

For, by corol. 2, art. 222, this body P is = - - - - $\frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{SP^2}$. But, by corol. 1, art. 221, $SG \cdot SO \cdot b = A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2$; therefore $P = \frac{SG \cdot SO}{SP} \cdot b$.

SCHOLIUM.

232. By the method of Fluxions, the centre of oscillation,

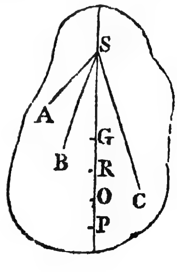
for a regular body, will be found from cor. 1. But for an irregular one; suspend it at the given point; and hang up also a simple pendulum of such a length, that making them both vibrate, they may keep time together. Then the length of the simple pendulum, is equal to the distance of the centre of oscillation of the body, below the point of suspension.

233. Or it will be still better found thus: Suspend the body very freely by the given point, and make it vibrate in very small arcs, counting the number of vibrations it makes in any time, as a minute, by a good stop watch; and let that number of vibrations made in a minute be called n: Then shall the distance of the centre of oscillation, be $so = \frac{140850}{nn}$ inches. For, the length of the pendulum vibrating seconds, or 60 times in a minute, being $39\frac{1}{8}$ inches; and the lengths of pendulums being reciprocally as the square of the number of vibrations made in the same time; therefore - - - -
$n^2 : 60^2 :: 39\frac{1}{8} : \frac{60^2 \times 39\frac{1}{8}}{nn} = \frac{140850}{nn}$: the length of the pendulum which vibrates n times in a minute, or the distance of the centre of oscillation below the axis of motion.

Or, $so = 39\frac{1}{8}\, t^2$, in inches, t being the time of one oscillation in a very small arc.

234. The foregoing determination of the point, into which all the matter of a body being collected, it shall oscillate in the same manner as before, only respects the case in which the body is put in motion by the gravity of its own particles, and the point is the centre of oscillation: but when the body is put in motion by some other extraneous force, instead of its gravity, and made to rotate instead of oscillate, then the point is different from the former, and is called the Centre of Gyration; which is determined in the following manner:

235. Prop. To determine the centre of gyration of a compound body or of a system of bodies.



Let R be the centre of gyration, or the point into which all the particles A, B, C, &c, being collected, it shall receive the same angular motion from a force f acting at P, as the whole system receives.

Now, by cor. 3, art. 228, the angular velocity generated in the system by the force f, is as $\frac{f \cdot SP}{A \cdot SA^2 + B \cdot SB^2 \,\&c}$,

and by the same, the angular velocity of the system placed in R, is $\frac{f \cdot SP}{(A + B + C \text{ \&c}) \cdot SR^2}$: then, by making these two expressions equal to each other, the equation gives $SR = \sqrt{\frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{A + B + C}}$, for the distance of the centre of gyration below the axis of motion.

236. *Corol.* 1. Because $A \cdot SA^2 + B \cdot SB^2$ &c $= SG \cdot SO \cdot W$, where G is the centre of gravity, O the centre of oscillation, and W the weight of the body A + B + C &c; therefore $SR^2 = SG \cdot SO$; that is, the distance of the centre of gyration, from the point of suspension, is a mean proportional between those of gravity and oscillation.

237. *Corol.* 2. If p denote any particle of a body W, at d distance from the axis of motion; then SR^2

$$= \frac{\text{sum of all the } pd^2}{\text{body W}}.$$

Or, if ρ be put for SR, the distance of the centre of gyration from the point of suspension, $W\rho^2 = A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2 +$ &c. = sum of all the pd^2.

SCHOLIUM.

238. By means of the theory of the centre of gyration, and the values of ρ thence deduced in the note to prop. 2. chap. xi. vol. iii. the phænomena of rotatory motion become connected with those of accelerating forces: for then, if a weight or other moving power P act at a radius r to give rotation to a body, weight W, and dist. of centre of gyration from axis of motion $= \rho$, we shall have for the accelerating force, the expression

$$f = \frac{Pr^2}{Pr^2 + W\rho^2};$$

and consequently for the space descended by the actuating weight or power P, in a given time t, we shall have the usual formula

$$s = \tfrac{1}{2} fgt^2,$$

introducing the above value of f.

239. For applications of these formulæ and their obvious modifications, as they are exceedingly useful in rotatory motions, the student may solve the following problems.

Problems illustrative of the Principle of the Centre of Gyration.

1. Suppose a cylinder that weighs 100lbs. to turn upon a horizontal axis, and imagine motion to be communicated by a weight of 10lbs. attached to a cord which coils upon the surface of the cylinder: how far will that weight descend in 10 seconds? Ans. 268·055 f.

2. Required the actuating weight such that when attached in the same way to the same cylinder, it shall descend $16\frac{1}{12}$ feet in 3 seconds. $\text{P} = \dfrac{\frac{1}{2}\text{SW}}{g\text{T}^2 - \text{S}} = 6\frac{1}{4}$.

3. Another cylinder, which weighs 200lbs. is actuated in like manner by a weight of 30lbs. How far will the weight descend in 6 seconds? Ans. 133·6 feet.

4. Suppose the actuating weight to be 30 pounds; and that it descends through 48 feet in 2 seconds, what is the weight of the cylinder? Ans. $20\frac{5}{12}$lbs.

5. Suppose a cylinder that weighs 20lbs. to have a weight of 30lbs. actuating it, by means of a cord coiled about the surface of the cylinder; what velocity will the descending weight have acquired at the end of the first second?

Ans. $24\frac{1}{8}$.

6. Of what weight will the axis be relieved in the case of the last example, when the system is completely in motion?

Ans. $22\frac{1}{2}$lbs.

7. A sphere, W, whose radius is three feet, and weight 500lbs. turns upon a horizontal axis, being put in motion by a weight of 20lbs. acting by means of a string that goes over a wheel whose radius is half a foot. How long will the weight, P, be in descending 50 feet? Ans. $33\frac{1}{2}''$.

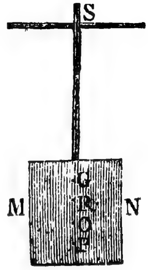
8. Of what weight will the axlo be relieved as soon as motion is commenced? Ans. $\frac{20}{361}$lbs.

9. If in example seventh the radius of the wheel be equal to that of the sphere, what ratio will the accelerating force bear to that of gravity?

10. A paraboloid, W, whose weight is 200lbs. and radius of base 20 inches, is put in motion upon a horizontal axis by a weight P of 15lbs. acting by a cord that passes over a wheel whose radius is 6 inches. After P has descended for 10 seconds, suppose it to reach a horizontal plane and cease to act, then how many revolutions would the paraboloid make in a minute?

BALLISTIC PENDULUM.

240. Prop. To explain the construction of the Ballistic Pendulum, and show its use in determining the velocity with which a cannon or other ball strikes it.



The ballistic pendulum is a heavy block of wood MN, suspended vertically by a strong horizontal iron axis at S, to which it is connected by a firm iron stem. This problem is the application of the preceding articles, and was invented by Mr. Robins, to determine the initial velocities of military projectiles; a circumstance very useful in that science; and it is the best method yet known for determining them with any degree of accuracy.

Let G, R, O be the centres of gravity, gyration, and oscillation, as determined by the foregoing propositions; and let P be the point where the ball strikes the face of the pendulum; the momentum of which, or the product of its weight and velocity, is expressed by the force f, acting at P, in the foregoing propositions. Now,

Put p = the whole weight of the pendulum,
b = the weight of the ball,
g = SG the distance of the centre of gravity,
o = SO the distance of the centre of oscillation,
r = SR = $\sqrt{go}$ the distance of centre of gyration,
i = SP the distance of the point of impact,
v = the velocity of the ball,
u = that of the point of impact P,
c = chord of the arc described by O.

By art. 235, if the mass p be placed all at R, the pendulum will receive the same motion from the blow in the point P: and as $\mathrm{SP}^2 : \mathrm{SR}^2 :: p : \frac{\mathrm{SR}^2}{\mathrm{SP}^2} \cdot p$ or $\frac{r^2}{i^2} p$ or $\frac{go}{ii} p$, (art. 236), the mass which being placed at P, the pendulum will still receive the same motion as before. Here then are two quantities of matter, namely, b and $\frac{go}{ii} p$, the former moving with the velocity v, and striking the latter at rest; to determine their common velocity u, with which they will jointly

proceed forward together after the stroke. In which case, by the law of the impact of non-elastic bodies, we have $\frac{go}{ii}p + b : b :: v : u$, and therefore $v = \frac{bii + gop}{bii}u$ the velocity of the ball in terms of u, the velocity of the point P, and the known dimensions and weights of the bodies.

But now to determine the value of u, we must have recourse to the angle through which the pendulum vibrates; for when the pendulum descends again to the vertical position, it will have acquired the same velocity with which it began to ascend, and by the laws of falling bodies, the velocity of the centre of oscillation is such as a heavy body would acquire by freely falling through the versed sine of the arc described by the same centre O. But the chord of that arc is c, and its radius is o; and, by the nature of the circle, the chord is a mean proportional between the versed sine and diameter, therefore $2o : c :: c : \frac{cc}{2o}$, the versed sine of the arc described by O. Then, by the laws of falling bodies $\sqrt{16\frac{1}{12}} : \sqrt{\frac{cc}{2o}} :: 32\frac{1}{6} : c\sqrt{\frac{2a}{o}}$, the velocity acquired by the point O in descending through the arc whose chord is c, where $a = 16\frac{1}{12}$ feet: and therefore $o : i :: c\sqrt{\frac{2a}{o}} : \frac{ci}{o}\sqrt{\frac{2a}{o}}$, which is the velocity u, of the point P.

Then, by substituting this value for u, the velocity of the ball, before found, becomes $v = \frac{bii + gop}{bio} \times c\sqrt{\frac{2a}{o}}$. So that the velocity of the ball is directly as the chord of the arc described by the pendulum in its vibration.

SCHOLIUM.

241. In the foregoing solution, the change in the centre of oscillation is omitted, which is caused by the ball lodging in the point P. But the allowance for that small change, and that of some other small quantities, may be seen in my Tracts, where all the circumstances of this method are treated at full length.

For an example in numbers, suppose the weights and dimensions to be as follow: namely,

$p = 570\text{lb.}$ Then

$b = 18\text{oz}\, 1\frac{1}{2}\text{dr} = 1{\cdot}131\text{lb.}$ $\quad \dfrac{bii + gop}{bio} \times c = \dfrac{1{\cdot}131 \times 94{\cdot}3^2 + 78\frac{1}{2} \times 84\frac{7}{9} \times 570}{1{\cdot}131 \times 94\frac{3}{10} \times 84\frac{7}{9}}$

$g = 78\frac{1}{2}$ inc. $\quad \times \dfrac{18{\cdot}73}{12} = 656{\cdot}56,$

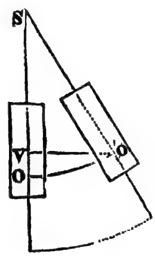
$o = 84\frac{7}{9}$ inc. $= 7{\cdot}065$ feet

$i = 94\frac{3}{10}$ inc. $\quad$ And $\sqrt{\dfrac{2a}{o}} = \sqrt{\dfrac{32\frac{1}{6}}{7{\cdot}065}} = \sqrt{\dfrac{193}{42{\cdot}39}} = 2{\cdot}1337.$

$c = 18{\cdot}73$ inc

Therefore $656{\cdot}56 \times 2{\cdot}1337$, or 1401 feet, is the velocity, per second, with which the ball moved when it struck the pendulum.

242. When the impact is made upon the centre of oscillation, the computation becomes simplified.



In that case, since the whole mass, p, of the pendulum, may be regarded as concentered at o, and the ball, b, strikes that point, we shall have $bv = (b + p)\, v'$; v being the velocity of the ball before the impact, and v' that of the ball and pendulum together, after the impact. Now, if the centre of oscillation o, after the blow, describes the arc oo′, before the motion is destroyed, the velocity v' will be equal to that acquired by falling through the versed sine vo, of the arc oo′ or angle s to the radius so. But, if the time t of a very minute oscillation of the pendulum be known or inferred from that in an ascertained arc (vol. iii. p. 357), we have (art. 233), so $= 39\frac{1}{8}t^2$ inches $= 3\frac{25}{96}t^2$ feet.

$$\text{Hence vo} = \text{so nat. versin s,}$$
$$= 3{\cdot}2604\tfrac{1}{6}t^2 \text{ versin s,}$$
$$\text{and (art. 154) } v' = \sqrt{(64\tfrac{1}{3} \times 3{\cdot}2604\tfrac{1}{6}t^2 \text{ versin s})}$$
$$= 14{\cdot}48286t \sqrt{\text{versin s}}.$$

$$\text{Conseq. } v = \frac{(b+p)v'}{b} = \frac{b+p}{b} \cdot 14{\cdot}48286t \sqrt{\text{versin s}}.$$

This mode of computation, with a slight and obvious change, applies to qu. 48 of the Practical Exercises in Natural Philosophy.

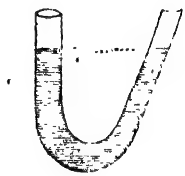
OF HYDROSTATICS.

243. Hydrostatics is the science which treats of the pressure, or weight, and equilibrium of water and other fluids, especially those that are non-elastic.

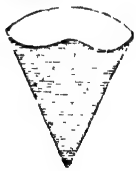
244. A fluid is elastic, when it can be reduced into a less bulk by compression, and which restores itself to its former bulk again when the pressure is removed; as air. And it is non-elastic, when it is not compressible or expansible, as water, &c.

245. Prop. If any part of a fluid be raised higher than the rest, by any force, and then left to itself; the higher parts will descend to the lower places, and the fluid will not rest, till its surface be quite even and level.

For, the parts of a fluid being easily moveable every way, the higher parts will descend by their superior gravity, and raise the lower parts, till the whole come to rest in a level or horizontal plane.



Corol. 1. Hence, water that communicates with other water, by means of a close canal or pipe, will stand at the same height in both places. Like as water in the two legs of a syphon.



Corol. 2. For the same reason, if a fluid gravitate towards a centre; it will dispose itself into a spherical figure, the centre of which is the centre of force. Like the sea in respect of the earth.

246. Prop. When a fluid is at rest in a vessel, the base of which is parallel to the horizon; equal parts of the base are equally pressed by the fluid.

For, on every equal part of this base there is an equal column of the fluid supported by it. And as all the columns are of equal height, by the last proposition they are of equal weight, and therefore they press the base equally; that is, equal parts of the base sustain an equal pressure.

Corol. 1. All parts of the fluid press equally at the same depth. For, if a plane parallel to the horizon be conceived to be drawn at that depth; then the pressure being the same in any part of that plane, by the proposition, therefore the parts of the fluid, instead of the plane, sustain the same pressure at the same depth.

Corol. 2. The pressure of the fluid at any depth, is as the depth of the fluid. For the pressure is as the weight, and the weight is as the height of the fluid.

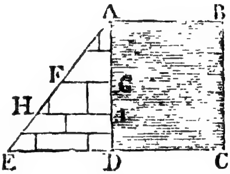
Corol. 3. The pressure of the fluid on any horizontal sur-

face or plane, is equal to the weight of a column of the fluid, whose base is equal to that plane, and altitude is its depth below the upper surface of the fluid.

247. Prop. When a fluid is pressed by its own weight, or by any other force; at any point it presses equally, in all directions whatever.

This arises from the nature of fluidity, by which it yields to any force in any direction. If it cannot recede from any force applied, it will press against other parts of the fluid in the direction of that force. And the pressure in all directions will be the same: for if it were less in any part, the fluid would move that way, till the pressure be equal every way.

Corol. 1. In a vessel containing a fluid; the pressure is the same against the bottom, as against the sides, or even upwards at the same depth.



Corol. 2. Hence, and from the last proposition, if ABCD be a vessel of water, and there be taken, in the base produced, DE, to represent the pressure at the bottom; joining AE, and drawing any parallels to the base, as FG, HI; then shall FG represent the pressure at the depth AG, and HI the pressure at the depth AI, and so on; because the parallels - FG, HI, ED, by sim. triangles, are as the depths AG, AI, AD: which are as the pressures, by the proposition.

And hence the sum of all the FG, HI, &c, or the area of the triangle ADE, is as the pressure against all the points G, I, &c, that is, against the line AD. But as every point in the line CD is pressed with a force as DE, and that thence the pressure on the whole line CD is as the rectangle ED . DC, while that against the side is as the triangle ADE or $\frac{1}{2}$DA . DE; therefore the pressure on the horizontal line DC, is to the pressure against the vertical line DA, as DC to $\frac{1}{2}$DA. And hence, if the vessel be an upright rectangular one, the pressure on the bottom, or whole weight of the fluid, is to the pressure against one side, as the base is to half that side. Therefore the weight of the fluid is to the pressure against all the four upright sides, as the base is to half the upright surface. And the same holds true also in any upright vessel, whatever the sides be, or in a cylindrical vessel. Or, in the

cylinder, the weight of the fluid is to the pressure against the upright surface, as the radius of the base is to double the altitude.

Also, when the rectangular prism becomes a cube, it appears that the weight of the fluid on the base, is double the pressure against one of the upright sides, or half the pressure against the whole upright surface.

Corol. 3. The pressure of a fluid against any upright surface, as the gate of a sluice or canal, is equal to half the weight of a column of the fluid whose base is equal to the surface pressed, and its altitude the same as the altitude of that surface. For the pressure on a horizontal base equal to the upright surface, is equal to that column; and the pressure on the upright surface, is but half that on the base, of the same area.

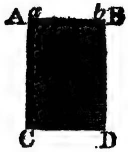
So that, if b denote the breadth, and d the depth of such a gate or upright surface; then the pressure against it, is equal to the weight of the fluid whose magnitude is $\frac{1}{2}bd^2 = \frac{1}{2}\text{AB} \cdot \text{AD}^2$. Hence, if the fluid be water, a cubic foot of which weighs 1000 ounces, or $62\frac{1}{2}$ pounds; and if the depth AD be 12 feet, the breadth AB 20 feet; then the content, or $\frac{1}{2}\text{AB} \cdot \text{AD}^2$, is 1440 feet; and the pressure is 1440000 ounces, or 90000 pounds, or $40\frac{1}{5}$ tons weight nearly.

248. PROP. The pressure of a fluid on a surface any way immersed in it, whether perpendicular, or horizontal, or oblique; is equal to the weight of a column of the fluid, whose base is equal to the surface pressed, and its altitude equal to the depth of the centre of gravity of the surface pressed below the top or surface of the fluid.

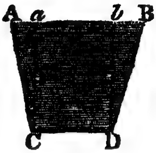
For, conceive the surface pressed to be divided into innumerable sections parallel to the horizon; and let s denote any one of those horizontal sections, also d its distance or depth below the top surface of the fluid. Then, by art. 246, *cor.* 3, the pressure of the fluid on the section is equal to the weight of ds; consequently the total pressure on the whole surface is equal to all the weights ds. But, if b denote the whole surface pressed, and g the depth of its centre of gravity below the top of the fluid; then, by art. 108, bg is equal to the sum of all the ds. Consequently the whole pressure of the fluid on the body or surface b, is equal to the weight of the bulk bg of the fluid, that is, of the column whose base is the given surface b, and its height is g the depth of the centre of gravity in the fluid.

249. PROP. The pressure of a fluid, on the base of the

vessel in which it is contained, is as the base and perpendicular altitude; whatever be the figure of the vessel that contains it.



If the sides of the base be upright, so that it be a prism of a uniform width throughout, then the case is evident; for then the base supports the whole fluid, and the pressure is just equal to the weight of the fluid.



But if the vessel be wider at top than bottom; then the bottom sustains, or is pressed by, only the part contained within the upright lines *a*C, *b*D; because the parts AC*a*, BD*b* are supported by the sides AC, BD; and those parts have no other effect on the part *ab*DC than keeping it in its position, by the lateral pressure against *a*C and *b*D, which does not alter its perpendicular pressure downwards. And thus the pressure on the bottom is less than the weight of the contained fluid.

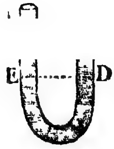
And if the vessel be widest at bottom; then the bottom is still pressed with a weight which is equal to that of the whole upright column *ab*DC. For, as the parts of the fluid are in equilibrio, all the parts have an equal pressure at the same depth; so that the parts within C*c* and *d*D press equally as those in *cd*, and therefore equally the same as if the sides of the vessel had gone upright to *a* and *b*, the defect of fluid in the parts AC*a* and BD*b* being exactly compensated by the downward pressure or resistance of the sides AC and BD against the contiguous fluid. And thus the pressure on the base may be made to exceed the weight of the contained fluid, in any proportion whatever.

So that, in general, be the vessels of any figure whatever, regular or irregular, upright or sloping, or variously wide and narrow in different parts, if the bases and perpendicular altitudes be but equal, the bases always sustain the same pressure. And as that pressure, in the regular upright vessel, is the whole column of the fluid, which is as the base and altitude; therefore the pressure in all figures is in that same ratio.

Corol. 1. Hence, when the heights are equal, the pressures are as the bases. And when the bases are equal, the pressure is as the height. But when both the heights and bases are equal, the pressures are equal in all, though their contents be ever so different.

Corol. 2. The pressure on the base of any vessel is the same as on that of a cylinder, of an equal base and height.

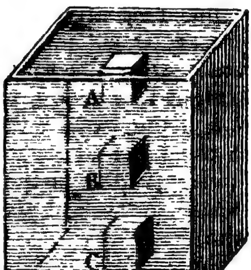
Corol. 3. If there be an inverted syphon, or bent tube, ABC, containing two different fluids CD, ABD, that balance each other, or rest in equilibrio; then their heights in the two legs, AE, CD, above the point of meeting, will be reciprocally as their densities.



For if they do not meet at the bottom, the part BD balances the part BE, and therefore the part CD balances the part AE; that is, the weight of CD is equal to the weight of AE. And as the surface at D is the same, where they act against each other, therefore AE : CD :: density of CD : density of AE.

So, if CD be water, and AE quicksilver, which is near 14 times heavier; then CD will be = 14AE; that is, if AE be 1 inch, CD will be 14 inches; if AE be 2 inches, CD will be 28 inches; and so on.

250. PROP. If a body be immersed in a fluid of the same density or specific gravity; it will rest in any place where it is put. But a body of greater density will sink; and one of a less density will rise to the top, and float.



The body being of the same density, or of the same weight with the like bulk of the fluid, will press the fluid under it, just as much as if its space were filled with the fluid itself. The pressure then all around it will be the same as if the fluid were in its place; consequently there is no force, neither upward nor downward, to put the body out of its place. And therefore it will remain wherever it is put.

But if the body be lighter; its pressure downward will be less than before, and less than the water upward at the same depth; therefore the greater force will overcome the less, and push the body upward to A.

And if the body be heavier than the fluid, the pressure downward will be greater than the fluid at the same depth; therefore the greater force will prevail, and carry the body down to the bottom at C.

Corol. 1. A body immersed in a fluid, loses as much weight, as an equal bulk of the fluid weighs. And the fluid

gains the same weight. Thus, if the body be of equal density with the fluid, it loses all its weight, and so requires no force but the fluid to sustain it. If it be heavier, its weight in the water will be only the difference between its own weight and the weight of the same bulk of water; and it requires a force to sustain it just equal to that difference. But if it be lighter, it requires a force equal to the same difference of weights to keep it from rising up in the fluid.

Corol. 2. The weights lost, by immerging the same body in different fluids, are as the specific gravities of the fluids. And bodies of equal weight, but different bulk, lose, in the same fluid, weights which are reciprocally as the specific gravities of the bodies, or directly as their bulks.

Corol. 3. The whole weight of a body which will float in a fluid, is equal to the weight of as much of the fluid, as the immersed part of the body displaces when it floats. For the pressure under the floating body, is just the same as so much of the fluid as is equal to the immersed part; and therefore the weights are the same.

Corol. 4. Hence the magnitude of the whole body, is to the magnitude of the part immersed, as the specific gravity of the fluid, is to that of the body. For, in bodies of equal weight, the densities, or specific gravities, are reciprocally as their magnitudes.

Corol. 5. And because, when the weight of a body taken in a fluid, is subtracted from its weight out of the fluid, the difference is the weight of an equal bulk of the fluid; this therefore is to its weight in the air, as the specific gravity of the fluid is to that of the body.

Therefore, if W be the weight of a body in air,
w its weight in water, or any fluid,
S the specific gravity of the body, and
s the specific gravity of the fluid;

then $\mathrm{W} - w : \mathrm{W} :: s : \mathrm{S}$, which proportion will give either of those specific gravities, the one from the other.

Thus $\mathrm{S} = \dfrac{\mathrm{W}}{\mathrm{W} - w} s$, the specific gravity of the body;

and $s = \dfrac{\mathrm{W} - w}{\mathrm{W}} \mathrm{S}$, the specific gravity of the fluid.

So that the specific gravities of bodies, are as their weights in the air directly, and their loss in the same fluid inversely.

Corol. 6. And hence, for two bodies connected together, or mixed together into one compound, of different specific

gravities, we have the following equations, denoting their weights and specific gravities, as below, viz.

H = weight of the heavier body in air, h = weight of the same in water, } s its spec. gravity;
L = weight of the lighter body in air, l = weight of the same in water, } s its spec. gravity;
C = weight of the compound in air, c = weight of the same in water, } f its spec. gravity;
w = the specific gravity of water. Then,

1st, $(H - h)\mathrm{s} = Hw$,
2d, $(L - l)s = Lw$,
3d, $(C - c)f = Cw$,
4th, $H + L = C$,
5th, $h + l = c$,
6th, $\frac{H}{\mathrm{s}} + \frac{L}{s} = \frac{C}{f}$

From which equations may be found any of the above quantities, in terms of the rest.

Thus, from one of the first three equations, is found the specific gravity of any body, as $s = \frac{Lw}{L - l}$, by dividing the absolute weight of the body by its loss in water, and multiplying by the specific gravity of water.

But if the body L be lighter than water; then l will be negative, and we must divide by $L + l$ instead of $L - l$, and to find l we must have recourse to the compound mass C; and because, from the 4th and 5th equations, $L - l = \overline{C - c} - \overline{H - h}$, therefore $s = \frac{Lw}{(C - c) - (H - h)}$; that is, divide the absolute weight of the light body, by the difference between the losses in water, of the compound and heavier body, and multiply by the specific gravity of water. Or thus, $s = \frac{\mathrm{s}fL}{C\mathrm{s} - Hf}$, as found from the last equation.

Also if it were required to find the quantities of two ingredients mixed in a compound, the 4th and 6th equations would give their values as follows, viz.

$$H = \frac{(f - s)\mathrm{s}}{(\mathrm{s} - s)f}C, \text{ and } L = \frac{(\mathrm{s} - f)s}{(\mathrm{s} - s)f}C,$$

the quantities of the two ingredients H and L, in the compound C. And so for any other demand.

PROP. To find the specific gravity of a body.

251. CASE I.—*When the body is heavier than water:* weigh it both in water and out of water, and take the difference, which will be the weight lost in water. Then, by corol. 6, art. 250, $s = \frac{Bw}{B - b}$, where B is the weight of the body out of

water, b its weight in water, s its specific gravity, and w the specific gravity of water. That is,

As the weight lost in water,
Is to the whole or absolute weight,
So is the specific gravity of water,
To the specific gravity of the body*.

EXAMPLE. If a piece of stone weigh 10lb, but in water only $6\frac{3}{4}$lb, required its specific gravity, that of water being 1000? Ans. 3077.

252. CASE II.—*When the body is lighter than water, so that it will not sink:* annex to it a piece of another body, heavier than water, so that the mass compounded of the two may sink together. Weigh the denser body and the compound mass, separately, both in water, and out of it; then find how much each loses in water, by subtracting its weight in water from its weight in air; and subtract the less of these remainders from the greater. Then say, by proportion,

As the last remainder,
Is to the weight of the light body in air,
So is the specific gravity of water,
To the specific gravity of the body.

That is, the specific gravity is $s = \frac{\text{L}w}{(\text{C} - c) - (\text{H} - h)}$, by cor. 6, art. 250.

EXAMPLE. Suppose a piece of elm weighs 15lb. in air; and that a piece of copper, which weighs 18lb. in air and 16lb. in water, is affixed to it, and that the compound weighs 6lb. in water; required the specific gravity of the elm? Ans. 600.

253. CASE III.—*For a fluid of any sort.*—Take a piece of a body of known specific gravity; weigh it both in and out of the fluid, finding the loss of weight by taking the difference of the two; then say,

As the whole or absolute weight,
Is to the loss of weight,
So is the specific gravity of the solid,
To the specific gravity of the fluid.

That is, the spec. grav. $w = \frac{\text{B} - b}{\text{B}}s$, by cor. 6, art, 250.

EXAMPLE. A piece of cast iron weighed 34·61 ounces in a fluid, and 40 ounces out of it; of what specific gravity is that fluid? Ans. 1000.

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* In the Lectures on Natural Philosophy, in the Royal Mil. Academy, Coates's Hydrostatic *steelyard* is employed for this purpose. It is an improvement upon the one described in No. 280 of Tilloch's Phil. Magazine.

254. Prop. To find the quantities of two ingredients in a given compound.

Take the three differences of every pair of the three specific gravities, namely, the specific gravities of the compound and each ingredient; and multiply each specific gravity by the difference of the other two. Then say, by proportion,

As the greatest product,
Is to the whole weight of the compound,
So is each of the other two products,
To the weights of the two ingredients.

That is, $\text{H} = \frac{(f - s)\text{S}}{(\text{S} - s)f}\text{C}$ = the one, and $\text{L} = \frac{(\text{S} - f)s}{(\text{S} - s)f}\text{C}$, the other, by cor. 6, art. 250.

Example. A composition of 112lb. being made of tin and copper, whose specific gravity is found to be 8784; required the quantity of each ingredient, the specific gravity of tin being 7320, and that of copper 9000?

Answer, there is 100lb. of copper and consequently 12lb. of tin, } in the composition.

SCHOLIUM.

255. The specific gravities of several sorts of matter, as found from experiments, are expressed by the numbers annexed to their names in the following Tables.

TABLES OF SPECIFIC GRAVITIES.

SOLIDS.

Platina - -	20,722	Bar Iron - -	7,788
Gold, pure, hammered	19,362	Pure Cornish Tin -	7,291
Guinea of George III.	17,629	Do. hardened -	7,299
Tungsten - -	17,600	Cast Iron - -	7,207
Mercury, at 32° Fahr.	13,598	Zinc - - -	6,862
Lead - -	11,352	Antimony - -	6,712
Palladium - -	11,300	Tellurium - -	6,115
Rhodium - -	11,000	Chromium - -	5,900
Virgin Silver -	10,744	Spar, heavy -	4,430
Shilling of George III.	10,534	Jargon of Ceylon -	4,416
Bismuth, molten -	9,822	Oriental Ruby -	4,283
Copper, wiredrawn	8,878	Sapphire, Oriental -	3,994
Red Copper, molten	8,788	Do. Brazilian -	3,131
Molybdena - -	8,611	Oriental Topaz -	4,019
Arsenic - -	8,308	Oriental Beryl -	3,549
Nickel, molten -	8,279	Diamond from 3,501 to	3,531
Uranium - -	8,100	English Flint-Glass	3,329
Steel - from 7,767 to	7,816	Tourmalin -	3,155
Cobalt, molten -	7,812	Asbestus	2,996

Marble, green, Campan.	2,742	Alabaster - -	1,874
——— Parian -	2,837	Alum - -	1,720
——— Norwegian -	2,728	Copal, opaque -	1,140
——— green, Egyptian	2,668	Sodium - -	973
Emerald - -	2,775	Oak, heart of, -	950
Pearl - - - -	2,752	Gunpowder, about -	937
Chalk, British -	2,784	Ice - - -	930
Jasper - -	2,710	Potassium - -	866
Coral - -	2,680	Beech - -	852
Rock Crystal -	2,653	Ash - -	845
English Pebble -	2,619	Apple-Tree - -	793
Limpid Feldspar -	2,564	Orange-Wood -	705
Glass, green - -	2,642	Pear-Tree - -	661
—— white - -	2,892	Linden-Tree -	604
—— bottle - -	2,733	Cypress - -	598
Porcelaine, China -	2,385	Cedar - -	561
——— Limoges	2,341	Fir - -	550
Native Sulphur -	2,033	Poplar - -	383
Ivory - -	1,917	Cork - -	240

LIQUIDS.

Sulphuric Acid -	1,841	Olive Oil - -	915
Nitrous Acid -	1,550	Muriatic Ether -	874
Water from the Dead Sea	1,240	Oil of Turpentine -	870
Nitric Acid - -	1,218	Liquid Bitumen -	848
Sea-Water - -	1,026	Alcohol, absolute -	792
Milk - - -	1,030	Sulphuric Ether -	716
Distilled Water -	1,000	Air at the Earth's Surface,	
Wine of Bourdeaux	994	about - -	$1\frac{2}{7}$
Burgundy Wine -	991		

*** Since a cubic foot of water at the temperature 40° Fahrenheit, weighs 1000 ounces avoirdupois, or $62\frac{1}{2}$ pounds, the numbers in the preceding Tables exhibit very nearly the respective weights of a cubic foot of the several substances tabulated.

256. PROP. To find the magnitude of any body, from its weight.

As the tabular specific gravity of the body,
Is to its weight in avoirdupois ounces,
So is one cubic foot, or 1728 cubic inches,
To its content in feet, or inches, respectively.

EXAM. 1. Required the content of an irregular block of green marble, which weighs 1 cwt, or 112lb?

Ans. 1160·6 cubic inches.

Exam. 2. How many cubic inches of gunpowder are there in 1lb. weight? Ans. $29\frac{1}{2}$ cubic inches nearly.

Exam. 3. How many cubic feet are there in a ton weight of dry oak? Spec. grav. 925. Ans. $38\frac{138}{185}$ cubic feet.

257. Prop. To find the weight of a body from its magnitude.

As one cubic foot, or 1728 cubic inches,
Is to the content of the body,
So is the tabular specific gravity,
To the weight of the body.

Exam. 1. Required the weight of a block of marble, whose length is 63 feet, and breadth and thickness each 12 feet; being the dimensions of one of the stones in the walls of Balbeck?

Ans. $683\frac{4}{15}$ ton, which is nearly equal to the burden of an East-India ship.

Exam. 2. What is the weight of 1 pint, ale measure, of gunpowder? Ans. 19 oz. nearly.

Exam. 3. What is the weight of a block of dry oak, which measures 10 feet in length, 3 feet broad, and $2\frac{1}{2}$ feet deep or thick? Ans. $4335\frac{15}{16}$lb.

BUOYANCY OF PONTOONS.

GENERAL SCHOLIUM.

258. The principles established in art. 250 have an interesting application to military men, in the use of pontoons, and the buoyancy by which they become serviceable in the construction of temporary bridges. When the dimensions, magnitude, and weight of a pontoon are known, that weight can readily be deducted from the weight of an equal bulk of water, and the remainder is evidently the weight which the pontoon will carry before it will sink.

Pontoons as usually constructed, are prisms whose vertical sections are equal trapezoids, as exhibited in the marginal figure.

Suppose $AB = L$
$CD = l$
$AI = KB = \frac{1}{2}(L - l) = \delta$
$CI = D$

Uniform width of the pontoon $= l$: all in feet and parts. Suppose also $CL = d$, depth of the part immersed; $w =$

weight in avoirdupois pounds of the water displaced; and $c = 62\frac{1}{2}$lbs. weight of a cubic foot of rain water. Then, by the following expressions, which are left for the student to investigate, d may be found when w and the rest are given, and w may be found when d and the rest are given; also the maximum value of w.

$$= bcd\left(l + \frac{d\delta}{\text{D}}\right)$$

2. w when a max. $= bc\text{D}\,(l + \delta) = \frac{1}{2}bc\text{D}\,(\text{L} + l)$

3. $d = \sqrt{\left[\frac{\text{D}}{\delta}\left(\frac{w}{bc} + \frac{l^2\text{D}}{4\delta}\right)\right]} - \frac{l\text{D}}{2\delta}$.

Ex. 1. Given AB $= 21\frac{1}{2}$ feet, CD $= 17\frac{1}{6}$ feet, CI $= 2\frac{1}{4}$ feet, $b = 4\frac{3}{4}$ feet. Required the weight of the pontoon and its load, when it is immersed to the depth CL of $1\frac{1}{2}$ feet.

Ans. $8287\frac{3}{4}$lbs. nearly.

Ex. 2. Suppose the weight of such a pontoon to be 900lbs. what is the greatest weight it will carry? Ans. $12014\frac{1}{16}$lbs.

Ex. 3. Suppose the weight of the above pontoon and its load to be 6000lbs. how deep will it sink in water?

Ans. $1{\cdot}08872f = 13{\cdot}064$ inches.

HYDRAULICS OR HYDRODYNAMICS.

259. Hydraulics or Hydrodynamics is that part of mechanical science which relates to the motion of fluids, and the forces with which they act upon bodies against which they strike, or which move in them.

This is a very extensive subject; but we shall here give only a few elementary propositions.

260. PROP. If a fluid run through a canal or river, or pipe of various widths, always filling it; the velocity of the fluid in different parts of it, AB, CD, will be reciprocally as the transverse sections in those parts.

That is, veloc. at A : veloc. at C : : CD : AB; where AB and CD denote, not the diameters at A and B, but the areas or sections there.

For, as the channel is always equally full, the quantity of water running through AB is equal to the quantity running through CD, in the same time; that is, the column through

AB is equal to the column through CD, in the same time; or AB × length of its column = CD × length of its column; therefore AB : CD : : length of column through CD : length of column through AB. But the uniform velocity of the water, is as the space run over, or length of the columns; therefore AB : CD : : velocity through CD : velocity through AB.

261. *Corol.* Hence, by observing the velocity at any place AB, the quantity of water discharged in a second, or any other time, will be found, namely, by multiplying the section AB by the velocity there.

But if the channel be not a close pipe or tunnel, kept always full, but an open canal or river; then the velocity in all parts of the section will not be the same, because the velocity towards the bottom and sides will be diminished by the friction against the bed or channel; and therefore a medium among the three ought to be taken. So, if the velocity at the top be - 100 feet per minute,
that at the bottom - 60
and that at the sides - 50

3)210 sum;

dividing their sum by 3, gives 70 for the mean velocity, which is to be multiplied by the section, to give the quantity discharged in a minute: and in many cases still greater accuracy will be necessary in determining the mean.

262. Prop. The velocity with which a fluid runs out by a hole in the bottom or side of a vessel, is equal to that which is generated by gravity through the height of the water above the hole; that is, the velocity of a heavy body acquired by falling freely through the height AB.

Divide the altitude AB into a great number of very small parts, each being 1, their number a, or a = the altitude AB.

Now, by art. 246, the pressure of the fluid against the hole B, by which the motion is generated, is equal to the weight of the column of fluid above it, that is the column whose height is AB or a, and base the area of the hole B. Therefore the pressure on the hole, or small part of the fluid 1, is to its weight, or the natural force of gravity, as a to 1. But, by art. 127, the velocities generated in the same body in any time, are as those forces; and because gravity generates the velocity 2 in descending through the small space 1, therefore $1 : a : : 2 : 2a$,

the velocity generated by the pressure of the column of fluid in the same time. But $2a$ is also, by corol. 1, art. 132, the velocity generated by gravity in descending through a or AB. That is, the velocity of the issuing water, is equal to that which is acquired by a body in falling through the height AB.

The same otherwise.

The momenta, or quantities of motion, generated in two given bodies, by the same force, acting during the same or an equal time, are equal. And the force in this case, is the weight of the superincumbent column of the fluid over the hole. Let then the one body to be moved, be that column itself, expressed by ah, where a denotes the altitude AB, and h the area of the hole; and the other body is the column of the fluid that runs out uniformly in one second suppose, with the middle or medium velocity of that interval of time, which is $\frac{1}{2}hv$, if v be the whole velocity required. Then the mass $\frac{1}{2}hv$, with the velocity v, gives the quantity of motion $\frac{1}{2}hv \times v$, or $\frac{1}{2}hv^2$, generated in one second, in the spouting water: also g, or $32\frac{1}{6}$ feet, is the velocity generated in the mass ah, during the same interval of one second; consequently $ah \times g$, or ahg, is the motion generated in the column ah in the same time of one second. But as these two momenta must be equal, this gives $\frac{1}{2}hv^2 = ahg$: hence then $v^2 = 2ag$, and $v = \sqrt{2ag}$, for the value of the velocity sought; which therefore is exactly the same as the velocity generated by the gravity in falling through the space a, or the whole height of the fluid.

For example, if the fluid were air, of the whole height of the atmosphere, supposed uniform, which is about $5\frac{1}{4}$ miles, or 27720 feet $= a$. Then $\sqrt{2ag} = 2\sqrt{(27720 \times 16\frac{1}{12})} = 1335$ feet $= v$ the velocity, that is, the velocity with which common air would rush into a vacuum.

263. *Corol.* 1. The velocity, and quantity run out, at different depths, are as the square roots of the depths. For the velocity acquired in falling through AB, is as $\sqrt{\text{AB}}$.

264. *Corol.* 2. The fluid spouts out with the same velocity, whether it be downward or upward, or sideways; because the pressure of fluids is the same in all directions, at the same depth. And therefore, if an adjutage be turned upward, the jet will ascend to the height of the surface of the water in the vessel. And this is confirmed by experience, by which it is found that jets really ascend nearly to the height of the reservoir, abating a small quantity only, for the friction against the sides, and some resistance from the air and from the oblique motion of the fluid in the hole.

265. *Corol.* 3. The quantity run out in any time, is equal to a column or prism, whose base is the area of the hole, and its length the space described in that time by the velocity acquired by falling through the altitude of the fluid. And the quantity is the same, whatever be the figure of the orifice, if it is of the same area.

Therefore, if a denote the altitude of the fluid,
and h the area of the orifice,
also $\frac{1}{2}g = 16\frac{1}{12}$ feet, or 193 inches;
then $2h\sqrt{\frac{1}{2}ag}$ will be the quantity of water discharged in a second of time; or nearly $8\frac{1}{48}h\sqrt{a}$ cubic feet, when a and h are taken in feet.

So, for example, if the height a be 25 inches, and the orifice $h = 1$ square inch; then $2h\sqrt{\frac{1}{2}ag} = 2\sqrt{25 \times 193} =$ 139 cubic inches, which is the quantity that would be discharged per second.

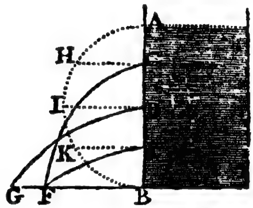
SCHOLIUM.

266. When the orifice is in the side of the vessel, then the velocity is different in the different parts of the hole, being less in the upper parts of it than in the lower. However, when the hole is but small, the difference is inconsiderable, and the altitude may be estimated from the centre to obtain the mean velocity. But when the orifice is pretty large, then the mean velocity is to be more accurately computed by other principles, given in the next proposition.

267. It is not to be expected that experiments, as to the quantity of water run out, will exactly agree with this theory, both on account of the resistance of the air, the resistance of the water against the sides of the orifice, and the oblique motion of the particles of the water in entering it. For, it is not merely the particles situated immediately in the column over the hole, which enter it and issue forth, as if that column only were in motion; but also particles from all the surrounding parts of the fluid, which is in a commotion quite around; and the particles thus entering the hole in all directions, strike against each other, and impede one another's motion: from which it happens, that it is the particles in the centre of the hole only that issue out with the whole velocity due to the entire height of the fluid, while the other particles towards the sides of the orifices pass out with decreased velocities; and hence the medium velocity through the orifice, is somewhat less than that of a single body only, urged with the same pressure of the superincumbent column of the fluid. And experiments on the quantity of water discharged through apertures, show that the quantity must

be diminished, by those causes, rather more than the fourth part, when the orifice is small, or such as to make the mean velocity nearly equal to that in a body falling through $\frac{1}{2}$ the height of the fluid above the orifice. If the velocity be taken as that due to the whole altitude above the orifice, then instead of the area of the orifice, the area of the contracted vein at a small distance from it must be taken. See Gregory's Mechanics and Bossut's Hydrodynamique.

268. Experiments have also been made on the extent to which the spout of water ranges on a horizontal plane, and compared with the theory, by calculating it as a projectile discharged with the velocity acquired by descending through the height of the fluid. For, when the aperture is in the side of the vessel, the fluid spouts out horizontally with a uniform velocity, which, combined with the perpendicular velocity from the action of gravity, causes the jet to form the curve of a parabola. Then the distances to which the jet will spout on the horizontal plane BG, will be as the roots of the rectangles of the segments AC . CB, AD . DB, AE . EB. For the spaces BF, BG, are as the times and horizontal velocities; but the velocity is as $\sqrt{AC}$; and the time of the fall, which is the same as the time of moving, is as $\sqrt{CB}$; therefore the distance BF is as $\sqrt{AC \cdot CB}$; and the distance BG as $\sqrt{AD \cdot DB}$. And hence, if two holes are made equidistant from the top and bottom, they will project the water to the same distance; for if AC = EB, then the rectangle AC . CB is equal the rectangle AE . EB: which makes BF the same for both. Or, if on the diameter AB a semicircle be described; then, because the squares of the ordinates CH, DI, EK are equal to the rectangles AC . EB, &c; therefore the distances BF, BG are as the ordinates CH, DI. And hence also it follows, that the projection from the middle point D will be farthest, for DI is the greatest ordinate.

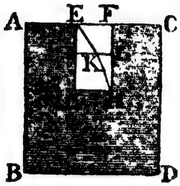


These are the *proportions* of the distances; but for the absolute distances, it will be thus. The velocity through any hole C, is such as will carry the water horizontally through a space equal to 2AC in the time of falling through AC: but, after quitting the hole, it describes a parabola, and comes to F in the time a body will fall through CB; and to find this distance, since the times are as the roots of

the spaces, therefore $\sqrt{AC} : \sqrt{CB} :: 2AC : 2\sqrt{AC \cdot CB} = 2CH = BF$, the space ranged on the horizontal plane. And the greatest range $BG = 2DI$, or $2AD$, or equal to AB.

And as these ranges answer very nearly to the experiments, this confirms the theory, as to the velocity assigned.

269. Prop. If a notch or slit EH in form of a parallelogram, be cut in the side of a vessel, full of water, AD; the quantity of water flowing through it, will be $\frac{2}{3}$ of the quantity flowing through an equal orifice, placed at the whole depth EG, or at the base GH, in the same time; it being supposed that the vessel is always kept full.



For the velocity at GH is to the velocity at IL, as $\sqrt{EG}$ to $\sqrt{EI}$; that is, as GH or IL to IK, the ordinate of a parabola EKH, whose axis is EG. Therefore the sum of the velocities at all the points I, is to as many times the velocity at G, as the sum of all the ordinates IK, to the sum of all the IL's; namely, as the area of the parabola EGH, is to the area EGHF; that is, the quantity running through the notch EH, is to the quantity running through an equal horizontal area placed at GH, as EGHKE, to EGHF, or as 2 to 3; the area of a parabola being $\frac{2}{3}$ of its circumscribing parallelogram.

Corol. 1. The mean velocity of the water in the notch, is equal to $\frac{2}{3}$ of that at GH.

Corol. 2. The quantity flowing through the hole IGHL, is to that which would flow through an equal orifice placed as low as GH, as the parabolic frustum IGHK, is to the rectangle IGHL. This appears from the demonstration.

OF PNEUMATICS.

270. Pneumatics is the science which treats of the properties of air, or elastic fluids.

271. Prop. Air is a fluid body; which surrounds the earth, and gravitates on all parts of its surface.

These properties of air are proved by experience.—That it is a fluid, is evident from its easily yielding to any the least force impressed on it, without making a sensible resistance.

But when it is moved briskly, by any means, as by a fan or a pair of bellows; or when any body is moved very briskly through it; in these cases we become sensible of it

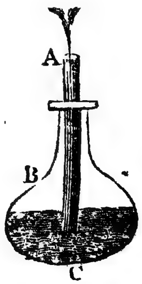
as a body, by the resistance it makes in such motions, and also by its impelling or blowing away any light substances. So that, being capable of resisting, or moving other bodies, by its impulse, it must itself be a body, and be heavy, like all other bodies, in proportion to the matter it contains; and therefore it will press on all bodies that are placed under it.

Also, as it is a fluid, it spreads itself all over on the earth; and, like other fluids, it gravitates and presses every where on the earth's surface.

272. The gravity and pressure of the air are also evident from many experiments. Thus, for instance, if water, or quicksilver, be poured into the tube ACE, and the air be suffered to press on it, in both ends of the tube, the fluid will rest at the same height in both legs: but if the air be drawn out of one end as E, by any means; then the air pressing on the other end A, will press down the fluid in this leg at B, and raise it up in the other to D, as much higher than at B, as the pressure of the air is equal to. From which it appears, not only that the air does really press, but also how much the intensity of that pressure is equal to. And this is the principle of the barometer.

273. PROP. The air is also an elastic fluid, being condensible and expansible: and the law it observes is this, that its density and elasticity are proportional to the force or weight which compresses it.

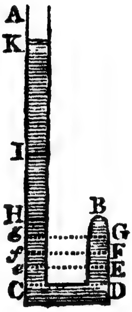
This property of the air is proved by many experiments. Thus, if the handle of a syringe be pushed inward, it will condense the inclosed air into less space, thereby showing its condensibility. But the included air, thus condensed, is felt to act strongly against the hand, resisting the force compressing it more and more; and, on withdrawing the hand, the handle is pushed back again to where it was at first. Which shows that the air is elastic.



274. Again, fill a strong bottle half full of water; then insert a small glass tube into it, putting its lower end down near to the bottom, and cementing it very close round the mouth of the bottle. Then, if air be strongly injected through the pipe, as by blowing with the mouth or otherwise, it will pass through the water from the lower end, ascending into the parts before occupied with air at B, and the whole mass of air become there condensed, because the

water is not compressible into a less space. But, on removing the force which injected the air at A, the water will begin to rise from thence in a jet, being pushed up the pipe by the increased elasticity of the air B, by which it presses on the surface of the water, and forces it through the pipe, till as much be expelled as there was air forced in; when the air at B will be reduced to the same density as at first, and, the balance being restored, the jet will cease.

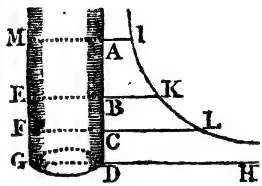
275. Likewise, if into a jar of water AB, be inverted an empty glass tumbler CD, or such-like, the mouth downward; the water will enter it, and partly fill it, but not near so high as the water in the jar, compressing and condensing the air into a less space in the upper parts C, and causing the glass to make a sensible resistance to the hand in pushing it down. Then, on removing the hand, the elasticity of the internal condensed air throws the glass up again. All these showing that the air is condensible and elastic.



276. Again, to show the relation of the elasticity to the condensation: take a long crooked glass tube, equally wide throughout, or at least in the part BD, and open at A, but close at the other end B. Pour in a little quicksilver at A, just to cover the bottom to the bend at CD, and to stop the communication between the external air and the air in BD. Then pour in more quicksilver, and mark the corresponding heights at which it stands in the two legs: so, when it rises to H in the open leg AC, let it rise to E in the close one, reducing its included air from the natural bulk BD to the contracted space BE, by the pressure of the column He; and when the quicksilver stands at I and K, in the open leg, let it rise to F and G in the other, reducing the air to the respective spaces BF, BG, by the weights of the columns If, Kg. Then it is always found, within moderate limits, that the condensations and elasticities are as the compressing weights and columns of the quicksilver, and the atmosphere together. So, if the natural bulk of the air BD be compressed into the spaces BE, BF, BG, which are $\frac{3}{4}$, $\frac{2}{4}$, $\frac{1}{4}$ of BD, or as the numbers 3, 2, 1; then the atmosphere, together with the corresponding columns He, If, Kg, are also found to be in the same proportion reciprocally, viz. as $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{1}$, or as the numbers 2, 3, 6. And then H$e = \frac{1}{3}$A, I$f =$ A,

and $Kg = 3A$; where A is the weight of the atmosphere. Which show that the condensations are directly as the compressing forces. And the elasticities are in the same ratio, since the columns in AC are sustained by the elasticities in BD.

From the foregoing principles may be deduced many useful remarks, as in the following corollaries, viz.



277. *Corol.* 1. The space in which any quantity of air is confined, is reciprocally as the force that compresses it. So, the forces which confine a quantity of air in the cylindrical spaces AG, BG, CG, are reciprocally as the same, or reciprocally as the heights AD, BD, CD. And therefore if to the two perpendicular lines DA, DH, as asymptotes, the hyperbola IKL be described, and the ordinates AI, BK, CL be drawn; then the forces which confine the air in the spaces AG, BG, CG, will be directly as the corresponding ordinates AI, BK, CL, since these are reciprocally as the abscisses AD, BD, CD, by the nature of the hyperbola.

Corol. 2. All the air near the earth is in a state of compression, by the weight of the incumbent atmosphere.

Corol. 3. The air is denser near the earth, than in high places; or denser at the foot of a mountain, than at the top of it. And the higher above the earth, the less dense it is.

Corol. 4. The spring or elasticity of the air, is equal to the weight of the atmosphere above it; and they will produce the same effects: since they always sustain and balance each other.

Corol. 5. If the density of the air be increased, preserving the same heat or temperature, its spring or elasticity is also increased, and in the same proportion.

Corol. 6. By the pressure and gravity of the atmosphere, on the surface of fluids, the fluids are made to rise in any pipes or vessels, when the spring or pressure within is decreased or taken off.

278. Prop. Heat increases the elasticity of the air, and cold diminishes it. Or, heat expands, and cold condenses the air.

This property is also proved by experience.

Thus, tie a bladder very close with some air in it; and lay it before the fire: then as it warms it will more and more distend the bladder, and at last burst it, if the heat be con-

tinued, and increased high enough. But if the bladder be removed from the fire, as it cools it will contract again, as before. And it was on this principle that the first air-balloons were made by Montgolfier: for, by heating the air within them, by a fire beneath, the hot air distends them to a size which occupies a space in the atmosphere, whose weight of common air exceeds that of the balloon.

Also, if a cup or glass, with a little air in it, be inverted into a vessel of water; and the whole be heated over the fire, or otherwise; the air in the top will expand till it fill the glass, and expel the water out of it; and part of the air itself will follow, by continuing or increasing the heat.

Many other experiments, to the same effect, might be adduced, all proving the properties mentioned in the proposition.

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279. So that, when the force of the elasticity of air is considered, regard must be had to its heat or temperature; the same quantity of air being more or less elastic, as its heat is more or less. And it has been found, by experiment, that the elasticity is increased by the 435th part, for each degree of heat, of which there are 180, between the freezing and boiling heat of water, in Fahrenheit's thermometer.

N. B. Water expands about the $\frac{3}{20000}$ part, with each degree of heat. (Sir Geo. Shuckburgh, Philos. Trans. 1777, p. 560, &c.)

Also, the

Spec. grav. of air	1·201 or $1\frac{1}{5}$	when the barom. is 29·5,	
water	1000	and the therm. is 55°	
mercury	13592	which are their mean heights in this country.	

Or thus, air	1·222 or $1\frac{2}{9}$	when the barom. is 30,
water	1000	and thermometer 55.
mercury	13600	

280. **Prop.** The weight or pressure of the atmosphere, on any base at the earth's surface, is equal to the weight of a column of quicksilver, of the same base, and the height of which is between 28 and 31 inches.

This is proved by the barometer, an instrument which measures the pressure of the air, and which is described below (art. 302). For, at some seasons, and in some places, the air sustains and balances a column of mercury, of about 28 inches: but at other times it balances a column of 29, or 30, or near 31 inches high; seldom in the extremes 28 or

31, but commonly about the means 29 or 30. This variation depends partly on the different degrees of heat in the air near the surface of the earth, and partly on the commotions and changes in the atmosphere, from winds and other causes, by which it is accumulated in some places, and depressed in others, being thereby rendered denser and heavier, or rarer and lighter; which changes in its state are almost continually happening in any one place. But the medium state is commonly about $29\frac{1}{2}$ or 30 inches.

281. *Corol.* 1. Hence the pressure of the atmosphere on every square inch at the earth's surface, at a medium, is very near 15 pounds avoirdupois, or rather $14\frac{3}{4}$ pounds. For, a cubic foot of mercury weighing 13600 ounces nearly, an inch of it will weigh 7·866 or almost 8 ounces, or nearly half a pound, which is the weight of the atmosphere for every inch of the barometer on a base of a square inch; and therefore 30 inches, or the medium height, weighs very near $14\frac{3}{4}$ pounds.

282. *Corol.* 2. Hence also the weight or pressure of the atmosphere, is equal to that of a column of water from 32 to 35 feet high, or on a medium 33 or 34 feet high. For, water and quicksilver are in weight nearly as 1 to 13·6; so that the atmosphere will balance a column of water 13·6 times as high as one of quicksilver; consequently

13·6 times 28 inches = 381 inches, or $31\frac{1}{4}$ feet,
13·6 times 29 inches = 394 inches, or $32\frac{5}{6}$ feet,
13·6 times 30 inches = 408 inches, or 34 feet,
13·6 times 31 inches = 422 inches, or $35\frac{1}{6}$ feet.

And hence a common sucking pump (art. 292) will not raise water higher than about 33 or 34 feet. And a siphon will not run, if the perpendicular height of the top of it be more than about 33 or 34 feet (art. 291).

283. *Corol.* 3. If the air were of the same uniform density at every height up to the top of the atmosphere, as at the surface of the earth; its height would be about $5\frac{1}{4}$ miles at a medium. For, the weights of the same bulk of air and water, are nearly as 1·222 to 1000; therefore as 1·222 : 1000 :: $33\frac{3}{4}$ feet : 27600 feet, or $5\frac{1}{4}$ miles nearly. And so high the atmosphere would be, if it were *homogeneous*, or all of uniform density, like water. But, instead of that, from its expansive and elastic quality, it becomes continually more and more rare, the farther above the earth, in a certain proportion, which will be treated of below, as also the method of measuring heights by the barometer, which depends on it.

284. *Corol.* 4. From this proposition and the last it follows, that the height is always the same, of a *homogeneous atmosphere* above any place, which shall be all of the uniform density with the air there, and of equal weight or pressure with the real height of the atmosphere above that place, whether it be at the same place, at different times, or at any different places or heights above the earth; and that height is always about $5\frac{1}{4}$ miles, or 27600 feet, as above found. For, as the density varies in exact proportion to the weight of the column, therefore it requires a column of the same height in all cases, to make the respective weights or pressures. Thus, if W and w be the weights of atmosphere above any places, D and d their densities, and H and h the heights of the uniform columns, of the same densities and weights; then $H \times D = W$, and $h \times d = w$; therefore $\frac{W}{D}$ or H is equal to $\frac{w}{d}$ or h: the temperature being the same.

285. Prop. With regard to the atmosphere, at different heights above the earth, this law obtains that when the heights increase in arithmetical progression, the densities decrease in geometrical progression.

Let the indefinite perpendicular line AP, erected on the earth, be conceived to be divided into a great number of very small equal parts, A, B, C, D, &c, forming so many thin strata of air in the atmosphere, all of different density, gradually decreasing from the greatest at A: then the density of the several strata A, B, C, D, &c, will be in geometrical progression decreasing.

For, as the strata A, B, C, &c, are all of equal thickness, the quantity of matter in each of them, is as the density there; but the density in any one, being as the compressing force, is as the weight or quantity of all the matter from that place upward to the top of the atmosphere; therefore the quantity of matter in each stratum, is also as the whole quantity from that place upward. Now, if from the whole weight at any place as B, the weight or quantity in the stratum B be subtracted, the remainder is the weight at the next stratum C; that is, from each weight subtracting a part which is proportional to itself, leaves the next weight; or, which is the same thing, from each density subtracting a part which is proportional to itself, leaves the next density. But

when any quantities are continually diminished by parts which are proportional to themselves, the remainders form a series of continued proportionals: consequently these densities are in geometrical progression.

Thus, if the first density be D, and from each be taken its nth part; there will then remain its $\frac{n-1}{n}$ part, or the $\frac{m}{n}$ part, putting m for $n-1$; and therefore the series of densities will be D, $\frac{m}{n}$D, $\frac{m^2}{n^2}$D, $\frac{m^3}{n^3}$D, $\frac{m^4}{n^4}$D, &c, the common ratio of the series being that of n to m.

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286. Because the terms of an arithmetical series, are proportional to the logarithms of the terms of a geometrical series: therefore different altitudes above the earth's surface, are as the logarithms of the densities, or of the weights of air, at those altitudes.

So that, if D denote the density at the altitude A,
and d - the density at the altitude a;
then A being as the log. of D, and a as the log. of d,

the dif. of alt. A $-a$ will be as the log. D $-$ log. d, or log. $\frac{D}{d}$.

And if A $=0$, or D the density at the surface of the earth; then any altitude above the surface a, is as the log. of $\frac{D}{d}$.

Or, in general, the log. of $\frac{D}{d}$ is as the altitude of the one place above the other, whether the lower place be at the surface of the earth, or any where else.

And from this property is derived the method of determining the heights of mountains and other eminences, by the barometer, which (art. 302) is an instrument that measures the pressure or density of the air at any place. For, by taking, with this instrument, the pressure or density, at the foot of a hill for instance, and again at the top of it, the difference of the logarithms of these two pressures, or the logarithm of their quotient, will be as the difference of altitude, or as the height of the hill; supposing the temperatures of the air to be the same at both places, and the gravity of air not altered by the different distances from the earth's centre.

287. But as this formula expresses only the relations between different altitudes with respect to their densities, re-

course must be had to some experiment, to obtain the real altitude which corresponds to any given density, or the density which corresponds to a given altitude. And there are various experiments by which this may be done. The first, and most natural, is that which results from the known specific gravity of air, with respect to the whole pressure of the atmosphere on the surface of the earth. Now, as the altitude a is always as log. $\frac{D}{d}$; assume h so that $a = h \times \log. \frac{D}{d}$, where h will be of one constant value for all altitudes; and to determine that value, let a case be taken in which we know the altitude a corresponding to a known density d; as for instance, take $a = 1$ foot, or 1 inch, or some such small altitude; then, because the density D may be measured by the pressure of the atmosphere, or the uniform column of 27600 feet, when the temperature is 55°; therefore 27600 feet will denote the density D at the lower place, and 27599 the less density d at 1 foot above it; consequently $1 = h \times \log. \frac{27600}{27599}$; which, by the nature of logarithms, is nearly $= h \times \frac{\cdot 43429448}{27600} = \frac{h}{63551}$ nearly; and hence $h = 63551$ feet; which gives, for any altitude in general, this theorem, viz. $a = 63551 \times \log. \frac{D}{d}$, or $= 63551 \times \log. \frac{M}{m}$ feet, or $10592 \times \log. \frac{M}{m}$ fathoms; where M is the column of mercury which is equal to the pressure or weight of the atmosphere at the bottom, and m that at the top of the altitude a; and where M and m may be taken in any measure, either feet or inches, &c.

288. Note, that this formula is adapted to the mean temperature of the air 55°. But, for every degree of temperature different from this, in the medium between the temperatures at the top and bottom of the altitude a, that altitude will vary by its 435th part; which must be added, when that medium exceeds 55°, otherwise subtracted.

Note, also, that a column of 30 inches of mercury varies its length by about the $\frac{1}{320}$ part of an inch for every degree of heat, or rather $\frac{1}{9600}$ of the whole volume.

289. But the formula may be rendered much more convenient for use, by reducing the factor 10592 to 10000, by changing the temperature proportionally from 55°; thus, as the diff. 592 is the 18th part of the whole factor 10592; and as 18 is the 24th part of 435; therefore the correspond-

ing change of temperature is 24°, which reduces the 55° to 31°. So that the formula is, $a = 10000 \times \log. \frac{M}{m}$ fathoms, when the temperature is 31 degrees; and for every degree above that, the result is to be increased by so many times its 435th part.

290. Taking, instead of the logarithms, the first term of the logarithmic series, we have $55000 . \frac{B - b}{B + b}$, for the altitude in feet: B and b, being the heights of the barometrical columns observed at the bottom and top of the hill. This formula is for the mean temperature 55°, and is easily remembered because the effective figures of the co-efficient are also 55. The reductions for any other temperature are the same as in the logarithmic rule.

EXAM. 1. To find the height of a hill when the pressure of the atmosphere is equal to 29·68 inches of mercury at the bottom, and 25·28 at the top; the mean temperature being 50°? Ans. 4378 feet, or 730 fathoms.

EXAM. 2. To find the height of a hill when the atmosphere weighs 29·45 inches of mercury at the bottom, and 26·82 at the top, the mean temperature being 33°?
Ans. 2385 feet, or 397½ fathoms.

EXAM. 3. At what altitude is the density of the atmosphere only the 4th part of what it is at the earth's surface?
Ans. 6020 fathoms.

By the weight and pressure of the atmosphere, the effect and operations of pneumatic engines may be accounted for, and explained; such as siphons, pumps, barometers, &c.; of which it will be proper here to give a brief description.

OF THE SIPHON.

291. THE Siphon, or Syphon, is any bent tube, having its two legs either of equal or of unequal length.

If it be filled with water, and then inverted, with the two open ends downward, and held level in that position; the water will remain suspended in it, if the two legs be equal. For the atmosphere will press equally on the surface of the water in each end, and support them, if they are not more than 34 feet high; and the legs being equal, the water in them is an exact

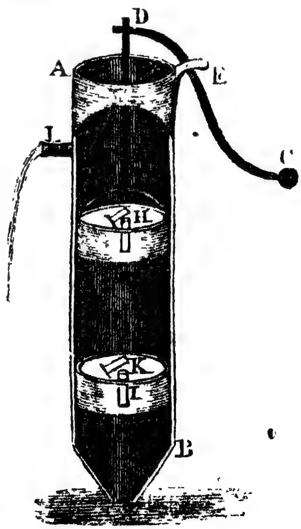
counterpoise by their equal weights; so that the one has no power to move more than the other; and they are both supported by the atmosphere.

But if now the siphon be a little inclined to one side, so that the orifice of one end be lower than that of the other; or if the legs be of unequal length, which is the same thing; then the equilibrium is destroyed, and the water will all descend out by the lower end, and rise up in the higher. For, the air pressing equally, but the two ends weighing unequally, a motion must commence where the power is greatest, and so continue till all the water has run out by the lower end. And if the shorter leg be immersed into a vessel of water, and the siphon be set running as above, it will continue to run till all the water be exhausted from the vessel, or at least as low as that end of the siphon. Or, it may be set running without filling the siphon as above, by only inverting it, with its shorter leg into the vessel of water; then, with the mouth applied to the lower orifice A, suck out the air; and the water will presently follow, being forced up into the siphon by the pressure of the air on the water in the vessel.

If a siphon be fixed in a vessel of water capable of rotation upon a vertical axis, and the orifice be lateral instead of at the bottom of the pipe, the reaction may be advantageously employed as a motive force. This is the principle of Mr. Busby's *Hydraulic Orrery*.

OF THE PUMP.

292.* THERE are three sorts of pumps: the Sucking, the Lifting, and the Forcing Pump. By the first, water can be raised only to about 33 feet, viz. by the pressure of the atmosphere; but by the others, to any height; but then they require more apparatus and power.



The annexed figure represents a common sucking pump. AB is the barrel of the pump, being a hollow cylinder, made of metal, and smooth within, or of wood for very common purposes. CD is the handle, moveable about the pin E, by moving the end C up and down. DF

an iron rod turning about a pin D, which connects it to the end of the handle. This rod is fixed to the piston, bucket, or sucker, FG, by which this is moved up and down within the barrel, which it must fit very tight and close, that no air or water may pass between the piston and the sides of the barrel; and for this purpose it is commonly armed with leather. The piston is made hollow, or it has a perforation through it, the orifice of which is covered by a valve H opening upwards. I is a plug firmly fixed in the lower part of the barrel, also perforated, and covered by a valve K opening upwards.

293. When the pump is first to be worked, and the water is below the plug I; raise the end C of the handle, then the piston descending, compresses the air in HI, which by its spring shuts fast the valve K, and pushes up the valve H, and so enters into the barrel above the piston. Then putting the end C of the handle down again, raises the piston or sucker, which lifts up with it the column of air above it, the external atmosphere by its pressure keeping the valve H shut: the air in the barrel being thus exhausted, or rarefied, is no longer a counterpoise to that which presses on the surface of the water in the well; this is forced up the pipe, and through the valve K, into the barrel of the pump. Then pushing the piston down again into this water, now in the barrel, its weight shuts the lower valve K, and its resistance forces up the valve of the piston, and enters the upper part of the barrel, above the piston. Then, the bucket being raised, lifts up with it the water which had passed above its valve, and it runs out by the cock L; and taking off the weight below it, the pressure of the external atmosphere on the water in the well again forces it up through the pipe and lower valve close to the piston, all the way as it ascends, thus keeping the barrel always full of water. And thus, by repeating the strokes of the piston, a continued discharge is made at the cock L.

294. There is a farther limitation of the operation, than that which relates to the 33 feet. If the elastic force of the air within the tube joined to the weight of water in the tube equal the pressure of the atmosphere, the water cannot rise in the pump. To prevent this, the product of the stroke of the piston into 33 must always exceed the square of half the greatest altitude of the piston above the surface of the water in the well. Otherwise diminish the diameter of the sucking-pipe proportionally.

OF THE AIR-PUMP.

295. NEARLY on the same principles as the water-pump, is the invention of the air-pump, by which the air is drawn out of any vessel, like as water is drawn out by the former. A brass barrel is bored and polished truly cylindrical, and exactly fitted with a turned piston, so that no air can pass by the sides of it, and furnished with a proper valve opening upward. Then, by lifting up the piston, the air in the close vessel below it follows the piston, and fills the barrel; and being thus diffused through a larger space than before, when it occupied the vessel or receiver only, but not the barrel, it is made rarer than it was before, in proportion as the capacity of the barrel and receiver together exceeds the receiver alone. Another stroke of the piston exhausts another barrel of this now rarer air, which again rarifies it in the same proportion as before. And so on, for any number of strokes of the piston, still exhausting in the same geometrical progression, of which the ratio is that which the capacity of the receiver and barrel together exceeds the receiver, till this is exhausted to any proposed degree, or as far as the nature of the machine is capable of performing; which happens when the elasticity of the included air is so far diminished, by rarefying, that it is too feeble to push up the valve of the piston, and escape.

296. From the nature of this exhausting, in geometrical progression, we may easily find how much the air in the receiver is rarefied by any number of strokes of the piston; or what number of such strokes is necessary, to exhaust the receiver to any given degree. Thus, if the capacity of the receiver and barrel together, be to that of the receiver alone, as c to r, and 1 denote the natural density of the air at first; then

$c : r :: 1 : \frac{r}{c}$, the density after 1 stroke of the piston,

$c : r :: \frac{r}{c} : \frac{r^2}{c^2}$, the density after two strokes,

$c : r :: \frac{r^2}{c^2} : \frac{r^3}{c^3}$, the density after three strokes,

&c, and $\frac{r^n}{c^n}$, the density after n strokes.

So, if the barrel be equal to $\frac{1}{4}$ of the receiver; then $c : r ::$

5 : 4; and $\frac{4^n}{5^n} = 0{\cdot}8^n$ is $= d$ the density after n turns. And if n be 20, then $0{\cdot}8^{20} = {\cdot}0115$ is the density of the included air after 20 strokes of the piston; which being the $86\frac{7}{10}$ part of 1, or the first density, it follows that the air is $86\frac{7}{10}$ times rarefied by the 20 strokes.

297. Or, if it were required to find the number of strokes necessary to rarefy the air any number of times; because $\frac{r^n}{c^n}$ is $=$ the proposed density d; therefore, taking the logarithms, $n \times \log. \frac{r}{c} = \log. d$, and $n = \frac{\log. d}{1.r - 1.c}$, the number of strokes required. So if r be $\frac{4}{5}$ of c, and it be required to rarefy the air 100 times: then $d = \frac{1}{100}$ or $\cdot 01$; and hence $n = \frac{\log. 100}{1.5 - 1.4} = 20\frac{3}{5}$ nearly. So that in $20\frac{3}{5}$ strokes the air will be rarefied 100 times.

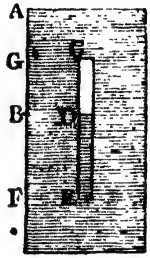
OF THE DIVING BELL AND CONDENSING MACHINE.

298. On the same principles too depend the operations and effect of the Condensing Engine, by which air may be condensed to any degree, instead of rarefied as in the air-pump. And, like as the air-pump rarefies the air, by extracting always one barrel of air after another; so, by this other machine, the air is condensed, by throwing in or adding always one barrel of air after another; which it is evident may be done by only turning the valves of the piston and barrel, that is, making them to open the contrary way, and working the piston in the same manner; so that, as they both open upward or outward in the air-pump, or rarefier, they will both open downward or inward in the condenser.

299. And on the same principles, namely, of the compression and elasticity of the air, depends the use of the Diving Bell, which is a large vessel, in which a person descends to the bottom of the sea, the open end of the vessel being downward; only in this case the air is not condensed by forcing more of it into the same space, as in the condensing engine; but by compressing the same quantity of air into a less space in the bell, by increasing always the force which compresses it.

300. If a vessel of any sort be inverted into water, and pushed or let down to any depth; then by the pressure of the water some of it will ascend into the vessel, but not so high as the water without, and will compress the air into less space, according to the difference between the heights of the internal and external water; and the density and elastic force of the air will be increased in the same proportion, as its space in the vessel is diminished.

So, if the tube CE be inverted, and pushed down into water, till the external water exceed the internal, by the height AB, and the air of the tube be reduced to the space CD; then that air is pressed both by a column of water of the height AB, and by the whole atmosphere which presses on the upper surface of the water; consequently the space CD is to the whole space CE, as the weight of the atmosphere, is to the weights both of the atmosphere and the column of water AB. So that, if AB be about 34 feet, which is equal to the force of the atmosphere, then CD will be equal to $\frac{1}{2}$CE; but if AB be double of that, or 68 feet, then CD will be $\frac{1}{3}$CE; and so on. And hence, by knowing the depth AF, to which the vessel is sunk, we can easily find the point D, to which the water will rise within it at any time. For let the weight of the atmosphere at that time be equal to that of 34 feet of water; also, let the depth AF be 20 feet, and the length of the tube CE 4 feet: then, putting the height of the internal water DE $= x$,



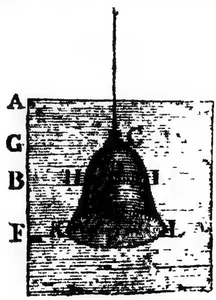
it is $34 + AB : 34 :: CE : CD$,

that is $34 + AF - DE : 34 :: CE : CE - DE$,

or $54 - x : 34 :: 4 : 4 - x$;

hence, multiplying extremes and means, $216 - 58x + x^2 = 136$, and the root is $x = \sqrt{2}$ very nearly $= 1{\cdot}414$ of a foot, or 17 inches nearly; being the height DE to which the water will rise within the tube.

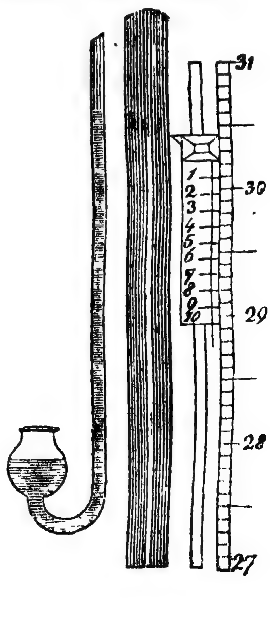
301. But if the vessel be not equally wide throughout, but of any other shape, as of a bell-like form, such as is used in diving; then the altitudes will not observe the proportion above, but the spaces or bulks only will accord with that proportion, namely, $34 + AB : 34 ::$ capacity CKL : capacity CHI, if it be common or fresh-water; and $33 + AB : 33 ::$ capacity CKL :



capacity CHI, if it be sea-water. From which proportion, the height DE may be found, when the nature and shape of the vessel or bell CKL are known.

OF THE BAROMETER.

302. THE Barometer is an instrument for measuring the pressure of the atmosphere, and elasticity of the air, at any time. It is commonly made of a glass tube, of near 3 feet long, close at one end, and filled with mercury. When the tube is full, by stopping the open end with the finger, then inverting the tube, and immersing that end with the finger into a bason of quicksilver, on removing the finger from the orifice, the fluid in the tube will descend into the bason, till what remains in the tube be of the same weight with a column of the atmosphere, which is commonly between 28 and 31 inches of quicksilver; and leaving an entire vacuum in the upper end of the tube above the mercury. For, as the upper end of the tube is quite void of air, there is no pressure downwards but from the column of quicksilver, and therefore that will be an exact balance to the counter pressure of the whole column of atmosphere, acting on the orifice of the tube by the quicksilver in the bason. The upper 3 inches of the tube, namely, from 28 to 31 inches, have a scale attached to them, divided into inches, tenths, and hundredths, for measuring the length of the column at all times, by observing which division of the scale the top of the quicksilver is opposite to; as it ascends and descends within these limits, according to the state of the atmosphere.



The weight of the quicksilver in the tube, above that in the bason, is at all times equal to the weight or pressure of the column of atmosphere above it, and of the same base with the tube; and hence the weight of it may at all times be computed; being nearly at the rate of half a pound avoirdupois for every inch of quicksilver in the tube, on every square inch of

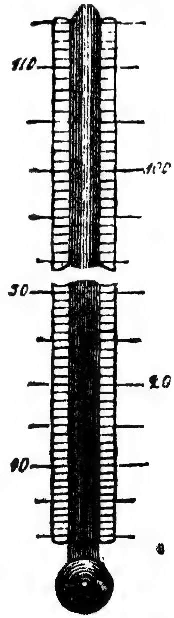
base; or more exactly it is $\frac{59}{120}$ of a pound on the square inch, for every inch in the altitude of the quicksilver weighs just $\frac{59}{120}$lb, or nearly $\frac{1}{2}$ a pound, in the mean temperature of 55° of heat. And consequently, when the barometer stands at 30 inches, or $2\frac{1}{2}$ feet high, which is nearly the medium or standard height, the whole pressure of the atmosphere is equal to $14\frac{3}{4}$ pounds, on every square inch of the base: and so in proportion for other heights.

Barometers are now constructed so as to be susceptible of convenient motion from place to place without derangement; thus facilitating the pneumatic method of determining the heights of hills, &c.

OF THE THERMOMETER.

303. The Thermometer is an instrument for measuring the temperature of the air, as to heat and cold.

It is found by experience, that all bodies expand by heat, and contract by cold: and since the expansion is, to a certain extent, uniform, the degrees of expansion become the measure of the degrees of heat. Fluids are more convenient for this purpose than solids: and quicksilver or mercury is now most commonly used for it. A very fine glass tube, having a pretty large hollow ball at the bottom, is filled about half way up with quicksilver: the whole being then heated very hot till the quicksilver rise quite to the top, the top is then hermetically sealed, so as perfectly to exclude all communication with the outward air. Then, in cooling, the quicksilver contracts, and consequently its surface descends in the tube, till it come to a certain point, correspondent to the temperature or heat of the air. And when the weather becomes warmer, the quicksilver expands, and its surface rises in the tube; and again contracts and descends when the weather becomes cooler. So that, by placing a scale of any divisions against the side of the tube, it will show the degrees of heat by the expansion and contraction of the quicksilver in the tube; observing at what division of the scale the top of



the quicksilver stands. The method of preparing the scale, as used in England, is thus:—Bring the thermometer into the temperature of freezing, by immersing the ball in water just freezing, or in ice just thawing, the latter is best, and mark the scale where the mercury then stands, for the point of freezing. Next, immerge it in boiling water; and the quicksilver will rise to a certain height in the tube; which mark also on the scale, for the boiling point, or the heat of boiling water. Then the distance between these two points, is divided into 180 equal divisions, or degrees; and the like equal degrees are also continued to any extent below the freezing point, and above the boiling point. The divisions are then numbered as follows, namely, at the freezing point is set the number 32, and consequently 212 at the boiling point; and all the other numbers in their order.

This division of the scale is commonly called *Fahrenheit's*. According to this division, 55 is at the mean temperature of the air in this country; and it is in this temperature, and in an atmosphere which sustains a column of 30 inches of quicksilver in the barometer, that all measures and specific gravities are taken, unless when otherwise mentioned; and in this temperature and pressure, the relative weights, or specific gravities of air, water, and quicksilver, are as

$1\frac{2}{9}$ for air,
1000 for water,
13600 for mercury; these also are the weights of a cubic foot of each, in avoirdupois ounces, in that state of the barometer and thermometer. For other states of the thermometer, each of these bodies expands or contracts according to the following rate, with each degree of heat, viz.

Air about - $\frac{1}{435}$ part of its bulk,
Water about $\frac{1}{6666}$ part of its bulk,
Mercury about $\frac{1}{9600}$ part of its bulk.

Another division is that of 100 equal degrees between the freezing and the boiling points, the 0 or *zero* being at the former. This is called the *centigrade* thermometer. It is now very common to put Fahrenheit's division on the left of the tube, and the centigrade division on the right.

ON THE MEASUREMENT OF ALTITUDES BY THE BAROMETER AND THERMOMETER.

304. From the principles laid down in arts. 286 to 289, concerning the measuring of altitudes by the barometer, and the foregoing descriptions of the barometer and thermometer,

we may now collect together the precepts for the practice of such measurements, which are as follow:

First. Observe the height of the barometer at the bottom of any height, or depth, intended to be measured; with the temperature of the quicksilver, by means of a thermometer attached to the barometer, and also the temperature of the air in the shade by a detached thermometer.

Secondly. Let the same thing be done also at the top of the said height or depth, and at the same time, or as near the same time as may be. And let those altitudes of barometer be reduced to the same temperature, if it be thought necessary, by correcting either the one or the other, that is, augment the height of the mercury in the colder temperature, or diminish that in the warmer, by its $\frac{1}{9600}$ part for every degree of difference of the two.

Thirdly. Take the difference of the common logarithms of the two heights of the barometer, corrected as above if necessary, cutting off 3 figures next the right hand for decimals, when the log-tables go to 7 figures, or cut off only 2 figures when the tables go to 6 places, and so on; or in general remove the decimal point 4 places more towards the right hand, those on the left hand being fathoms in whole numbers.

Fourthly. Correct the number last found for the difference of temperature of the air, as follows: take half the sum of the two temperatures, for the mean one; and for every degree which this differs from the temperature 31°, take so many times the $\frac{1}{435}$ part of the fathoms above found, and add them if the mean temperature be above 31°, but subtract them if the mean temperature be below 31°; and the sum or difference will be the true altitude in fathoms: or, being multiplied by 6, it will be the altitude in feet.

EXAM. 1. Let the state of the barometers and thermometers be as follows; to find the altitude, viz.

Barom.	Thermom. attach.	Thermom. detach.	
Lower 29·68	57	57	Ans. the alt. is
Upper 25·28	43	42	$719\frac{1}{3}$ fathoms.

EXAM. 2. To find the altitude, when the state of the barometers and thermometers is as follows, viz.

Barom.	Thermom. attach.	Thermom. detach.	
Lower 29·45	38	31	Ans. the alt. is $409\frac{8}{13}$ fathoms,
Upper 26·82	41	35	or 2458 feet.

This is a highly useful method within certain limits; but is by no means susceptible of that degree of accuracy which many have imputed to it.

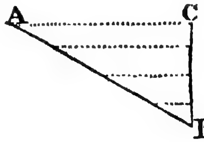
ON THE RESISTANCE OF FLUIDS, WITH THEIR FORCES AND ACTIONS ON BODIES.

305. PROP. If any body move through a fluid at rest, or the fluid move against the body at rest; the force or resistance of the fluid against the body, will be as the square of the velocity and the density of the fluid. That is, $R \propto dv^2$.

For, the force or resistance is as the quantity of matter or particles struck, and the velocity with which they are struck. But the quantity or number of particles struck in any time, are as the velocity and the density of the fluid. Therefore the resistance, or force of the fluid, is as the density and square of the velocity.

306. *Corol.* 1. The resistance to any plane, is also more or less, as the plane is greater or less; and therefore the resistance on any plane, is as the area of the plane a, the density of the medium, and the square of the velocity. That is, $R \propto adv^2$.

307. *Corol.* 2. If the motion be not perpendicular, but oblique to the plane, or to the face of the body; then the resistance, in the direction of motion, will be diminished in the triplicate ratio of radius to the sine of the angle of inclination of the plane to the direction of the motion, or as the cube of radius to the cube of the sine of that angle. So that $R \propto adv^2s^3$, putting $1 =$ radius, and $s =$ sine of the angle of inclination CAB.



For, if AB be the plane, AC the direction of motion, and BC perpendicular to AC; then no more particles meet the plane than what meet the perpendicular BC, and therefore their number is diminished as AB to BG or as 1 to s. But the force to each particle, striking the plane obliquely in the direction CA, is also diminished as AB to BC, or as 1 to s; therefore the resistance, which is perpendicular to the face of the plane is as 1^2 to s^2. But again, this resistance in the direction perpendicular to the face of the plane, is to that in the direction AC, by the parallelogram of forces, as AB to BC, or as 1 to s. Conse-

quently, on all these accounts, the resistance to the plane when moving perpendicular to its face, is to that when moving obliquely, as 1^3 to s^3, or 1 to s^3. That is, the resistance in the direction of the motion, is diminished as 1 to s^3, or in the triplicate ratio of radius to the sine of inclination.

308. PROP. The real resistance to a plane, from a fluid acting in a direction perpendicular to its face, is equal to the weight of a column of the fluid, whose base is the plane, and altitude equal to that which is due to the velocity of the motion, or through which a heavy body must fall to acquire that velocity.

The resistance to the plane moving through a fluid, is the same as the force of the fluid in motion with the same velocity, on the plane at rest. But the force of the fluid in motion, is equal to the weight or pressure which generates that motion; and this is equal to the weight or pressure of a column of the fluid, whose base is the area of the plane, and its altitude that which is due to the velocity.

309. *Corol.* 1. If a denote the area of the plane, v the velocity, n the density or specific gravity of the fluid, and $\frac{1}{2}g = 16\frac{1}{12}$ feet, or 193 inches. Then, the altitude due to the velocity v being $\frac{v^2}{2g}$, therefore $a \times n \times \frac{v^2}{2g} = \frac{anv^2}{2g}$ will be the whole resistance, or motive force R.

310. *Corol.* 2. If the direction of motion be not perpendicular to the face of the plane, but oblique to it, in any angle, whose sine is s. Then the resistance to the plane will be $\frac{anv^2s^3}{2g}$.

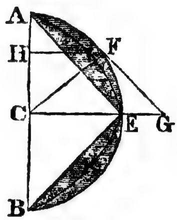
311. *Corol.* 3. Also, if w denote the weight of the body, whose plane face a is resisted by the absolute force R; then the retarding force f, or $\frac{\text{R}}{w}$ will be $\frac{anv^2s^3}{2gw}$.

312. *Corol.* 4. And if the body be a cylinder, whose face or end is a, and radius r, moving in the direction of its axis; because then $s = 1$, and $a = \pi r^2$, where $\pi = 3{\cdot}141593$; then $\frac{\pi nv^2r^2}{2g}$ will be the resisting force R, and $\frac{\pi nv^2r^2}{2gw}$ the retarding force f.

313. *Corol.* 5. This is the value of the resistance when the end of the cylinder is a plane perpendicular to its axis, or to the direction of motion. But were its face an elliptic

section, or a conical surface, or any other figure every where equally inclined to the axis, or direction of motion, the sine of inclination being s; then, the number of particles of the fluid striking the face being still the same, but the force of each, opposed to the direction of motion, diminished in the duplicate ratio of radius to the sine of inclination, the resisting force R would be $\frac{\pi n r^2 v^2 s^2}{2g}$.

314. PROP. The resistance to a sphere moving through a fluid, is but half the resistance to its great circle, or to the end of a cylinder of the same diameter, moving with an equal velocity.



Let AFEB be half the sphere, moving in the direction CEG. Describe the paraboloid AIEKB on the same base. Let any particle of the medium meet the semicircle in F, to which draw the tangent FG, the radius FC, and the ordinate FIH. Then the force of any particle on the surface at F, is to its force on the base at H, as the square of the sine of the angle G, or its equal the angle FCH, to the square of radius, that is, as HF^2 to CF^2. Therefore the force of all the particles, or the whole fluid, on the whole surface, is to its force on the circle of the base, as all the HF^2 to as many times CF^2. But CF^2 is $= CA^2 = AC \cdot CB$, and $HF^2 = AH \cdot HB$ by the nature of the circle: also, $AH \cdot HB : AC \cdot CB :: HI : CE$ by th. 2, parabola; consequently the force on the spherical surface is to the force on its circular base, as all the HI's to as many CE's, that is, as the content of the paraboloid to the content of its circumscribed cylinder, namely, as 1 to 2 (th. 18, parab.)

315. *Corol.* Hence, the resistance to the sphere is $R = \frac{\pi n v^2 r^2}{4g}$, being the half of that of a cylinder of the same diameter. For example, a 9lb. iron ball, whose diameter is 4 inches, when moving through the air with a velocity of 1600 feet per second, would meet a resistance which is equal to a weight of $132\frac{1}{3}$lb. over and above the pressure of the atmosphere, for want of the counterpoise behind the ball.

OF THE WEIGHT AND DIMENSIONS OF BALLS AND SHELLS.

THE weight and dimensions of Balls and Shells might be found from the problems given under the head of specific gravity. But they may be found still easier by means of the experimented weight of a ball of a given size, from the known proportion of similar figures, namely, as the cubes of their diameters, or like linear dimensions.

PROBLEM I.

To find the Weight of an Iron Ball, from its Diameter.

An iron ball of 4 inches diameter weighs 9lb. and the weights being as the cubes of the diameters, it will be, as 64 (which is the cube of 4) is to 9 its weight, so is the cube of the diameter of any other ball, to its weight. Or take $\frac{9}{64}$ of the cube of the diameter, for the weight. Or, take $\frac{1}{8}$ of the cube of the diameter, and $\frac{1}{8}$ of that again, and add the two together, for the weight.

EXAM. 1. The diameter of an iron shot being 6·7 inches, required its weight? Ans. 42·294lb.

EXAM. 2. What is the weight of an iron ball, whose diameter is 5·54 inches? Ans. 24lb. nearly.

PROBLEM II.

To find the Weight of a Leaden Ball.

A leaden ball of 1 inch diameter weighs $\frac{3}{14}$ of a lb.; therefore as the cube of 1 is to $\frac{3}{14}$, or as 14 is to 3, so is the cube of the diameter of a leaden ball, to its weight. Or, take $\frac{3}{14}$ of the cube of the diameter, for the weight, nearly.

EXAM. 1. Required the weight of a leaden ball of 6·6 inches diameter? Ans. 61·606lb.

EXAM. 2. What is the weight of a leaden ball of 5·30 inches diameter? Ans. 32lb. nearly.

EXAM. 3. How many shot, each $\frac{1}{40}$ of an inch diameter, may be made out of 10lb. of lead? Ans. 2986667.

PROBLEM III.

To find the Diameter of an Iron Ball.

Multiply the weight by $7\frac{1}{9}$, and the cube root of the product will be the diameter.

EXAM. 1. Required the diameter of a 42lb. iron ball?
Ans. 6·685 inches.

EXAM. 2. What is the diameter of a 24lb. iron ball?
Ans. 5·54 inches.

PROBLEM IV.

To find the Diameter of a Leaden Ball.

Multiply the weight by 14, and divide the product by 3; then the cube root of the quotient will be the diameter.

EXAM. 1. Required the diameter of a 64lb. leaden ball?
Ans. 6·684 inches.

EXAM. 2. What is the diameter of an 8lb. leaden ball?
Ans. 3·343 inches.

PROBLEM V.

To find the Weight of an Iron Shell.

Take $\frac{9}{64}$ of the difference of the cubes of the external and internal diameter, for the weight of the shell.

That is, from the cube of the external diameter, take the cube of the internal diameter, multiply the remainder by 9, and divide the product by 64.

EXAM. 1. The outside diameter of an iron shell being 12·8, and the inside diameter 9·1 inches; required its weight?
Ans. 188·941lb.

EXAM. 2. What is the weight of an iron shell, whose external and internal diameters are 9·8 and 7 inches?
Ans. $84\frac{1}{8}$lb.

PROBLEM VI.

To find how much Powder will fill a Shell.

Divide the cube of the internal diameter, in inches, by 57·3, for the lbs. of powder*.

* This and the following are only approximative rules, founded upon the supposition that, at a medium, 30 cubic inches of gunpowder weigh a pound. Of 18 different kinds of gunpowder used in the Royal Laboratory, Woolwich, the weights vary from 58lb. $\frac{1}{4}$oz. to 49lb. 13oz. per cubic foot, and the specific gravities, consequently, from 929 to 727. The specific gravity of French gunpowder usually lies between narrower limits; viz. those of 944 and 897.

EXAM. 1. How much powder will fill the shell whose internal diameter is 9·1 inches? Ans. $13\frac{2}{13}$lb. nearly.

EXAM. 2. How much powder will fill a shell whose internal diameter is 7 inches? Ans. 6lb.

PROBLEM VII.

To find how much Powder will fill a Rectangular Box.

Find the content of the box in inches, by multiplying the length, breadth, and depth all together. Then divide by 30 for the pounds of powder.

EXAM. 1. Required what quantity of powder will fill a box, the length being 15 inches, the breadth 12, and the depth 10 inches? Ans. 60lb.

EXAM. 2. How much powder will fill a cubical box whose side is 12 inches? Ans. $57\frac{3}{5}$lb.

PROBLEM VIII.

To find how much Powder will fill a Cylinder.

Multiply the square of the diameter by the length, then divide by 38·2 for the pounds of powder.

EXAM. 1. How much powder will the cylinder hold, whose diameter is 10 inches, and length 20 inches? Ans. $52\frac{1}{3}$lb. nearly.

EXAM. 2. How much powder can be contained in the cylinder whose diameter is 4 inches, and length 12 inches? Ans. $5\frac{5}{191}$lb.

PROBLEM IX.

To find the Size of a Shell to contain a given Weight of Powder.

Multiply the pounds of powder by 57·3, and the cube root of the product will be the diameter in inches.

EXAM. 1. What is the diameter of a shell that will hold $13\frac{1}{6}$lb. of powder? Ans. 9·1 inches.

EXAM. 2. What is the diameter of a shell to contain 6lb. of powder? Ans. 7 inches.

PROBLEM X.

To find the Size of a Cubical Box, to contain a given Weight of Powder.

Multiply the weight in pounds by 30, and the cube root of the product will be the side of the box in inches.

Exam. 1. Required the side of a cubical box, to hold 50lb. of gunpowder? Ans. 11·44 inches.

Exam. 2. Required the side of a cubical box, to hold 400lb. of gunpowder? Ans. 22·89 inches.

PROBLEM XI.

To find what Length of a Cylinder will be filled by a given Weight of Gunpowder.

Multiply the weight in pounds by 38·2, and divide the product by the square of the diameter in inches, for the length.

Exam. 1. What length of a 36-pounder gun, of $6\frac{2}{3}$ inches diameter, will be filled with 12lb. of gunpowder?
Ans. 10·314 inches.

Exam. 2. What length of a cylinder, of 8 inches diameter, may be filled with 20lb. of powder? Ans. $11\frac{15}{16}$ inches.

OF THE PILING OF BALLS AND SHELLS.

Iron Balls and Shells are commonly piled by horizontal courses, either in a pyramidical or in a wedge-like form; the base being either an equilateral triangle, or a square, or a rectangle. In the triangle and square, the pile finishes in a single ball; but in the rectangle, it finishes in a single row of balls, like an edge.

In triangular and square piles, the number of horizontal rows, or courses, is always equal to the number of balls in one side of the bottom row. And in rectangular piles, the number of rows is equal to the number of balls in the breadth of the bottow row. Also, the number in the top row, or edge, is one more than the difference between the length and breadth of the bottom row. A rule or two on this subject have been given in the first volume: the substance of them is repeated here, with a few additional rules.

PROBLEM I.

To find the Number of Balls in a Triangular Pile.

Multiply continually together the number of balls in one side of the bottom row, and that number increased by 1, also the same number increased by 2; then $\frac{1}{6}$ of the last product will be the answer.

That is, $\frac{1}{6}n \,.\, (n+1) \,.\, (n+2)$ is the nnmber or sum, where n is the number in the bottom row.

EXAM. 1. Required the number of balls in a triangular pile, each side of the base containing 30 balls? Ans. 4960.

EXAM. 2. How many balls are in the triangular pile, each side of the base containing 20? Ans. 1540.

PROBLEM II.

To find the Number of Balls in a Square Pile.

Multiply continually together the number in one side of the bottom course, that number increased by 1, and double the same number increased by 1; then $\frac{1}{6}$ of the last product will be the answer.

That is, $\frac{1}{6} n \,.\, (n+1) \,.\, (2n+1)$ is the number.

EXAM. 1. How many balls are in a square pile of 30 rows? Ans. 9455.

EXAM. 2. How many balls are in a square pile of 20 rows? Ans. 2870.

PROBLEM III.

To find the Number of Balls in a Rectangular Pile.

From 3 times the number in the length of the base row, subtract one less than the breadth of the same, multiply the remainder by the same breadth, and the product by one more than the same; and divide by 6 for the answer.

That is, $\frac{1}{6} b \,.\, (b+1) \,.\, (3l - b + 1)$ is the number; where l is the length, and b the breadth of the lowest course.

Note.—In all the piles the breadth of the bottom is equal to the number of courses. And in the oblong or rectangular pile, the top row is one more than the difference between the length and breadth of the bottom.

EXAM. 1. Required the number of balls in a rectangular pile, the length and breadth of the base row being 46 and 15? Ans. 4960.

EXAM. 2. How many shot are in a rectangular complete pile, the length of the bottom course being 59, and its breadth 20? Ans. 11060.

PROBLEM IV.

To find the Number of Balls in an Incomplete Pile.

From the number in the whole pile, considered as com-

plete, subtract the number in the upper pile which is wanting at the top, both computed by the rule for their proper form; and the remainder will be the number in the frustum, or incomplete pile.

Exam. 1. To find the number of shot in the incomplete triangular pile, one side of the bottom course being 40, and the top course 20? Ans. 10150.

Exam. 2. How many shot are in the incomplete triangular pile, the side of the base being 24, and of the top 8? Ans. 2516.

Exam. 3. How many balls are in the incomplete square pile, the side of the base being 24, and of the top 8? Ans. 4760.

Exam. 4. How many shot are in the incomplete rectangular pile, of 12 courses, the length and breadth of the base being 40 and 20? Ans. 6146.

OF DISTANCES BY THE VELOCITY OF SOUND.

From various experiments recently made, with great care, by the present editor of this volume, it has been found that sound flies through the air uniformly at the rate of about 1110 feet per second, when the air is quiescent, and at a medium temperature. At the temperature of freezing, or a little below, the velocity is 1100 feet; at the temperature of 75°, on Fahrenheit's thermometer, the velocity is about 1120. The approximate velocity under different temperatures may be found, by adding to 1100, *half a foot*, for every degree, on Fahrenheit's thermometer, above the freezing point. The mean velocity may be taken at 370 yards per second; or a mile in $4\frac{7}{9}$ seconds.

Hence, multiplying any time employed by sound in moving, by 370, will give the corresponding space in yards. Or, dividing any space in yards by 370, will give the time which sound will occupy in passing uniformly over that space.

If the wind blow briskly, as at the rate of from 20 to 60 feet per second, in the direction in which the sound moves, the velocity of the sound will be proportionably augmented: if the direction of the wind is opposed to that of the sound, the difference of their velocities must be employed.

Note.—The time for the passage of sound in the interval between seeing the flash of a gun, or lightning, and hearing

the report, may be observed by a watch, or a small pendulum. Or, it may be observed by the beats of the pulse in the wrist, counting, on an average, about 70 to a minute for persons in moderate health, or $5\frac{5}{9}$ pulsations to a mile; and more or less according to circumstances.

Exam. 1. After observing a flash of lightning, it was 12 seconds before the thunder was heard; required the distance of the cloud from whence it came? Ans. 2·52 miles.

Exam. 2. How long, after firing the Tower guns, may the report be heard at Shooter's-Hill, supposing the distance to be 8 miles in a straight line? Ans. $38\frac{2}{9}$ seconds.

Exam. 3. After observing the firing of a large cannon at a distance, it was 7 seconds before the report was heard; what was its distance? Ans. 1·47 mile.

Exam. 4. Perceiving a man at a distance hewing down a tree with an axe, I remarked that 4 of my pulsations passed between seeing him strike and hearing the report of the blow; what was the distance between us, allowing 70 pulses to a minute?

Exam. 5. How far off was the cloud from which thunder issued, whose report was 5 pulsations after the flash of lightning; counting 75 to a minute?

Exam. 6. If I see the flash of a cannon, fired by a ship in distress at sea, and hear the report 33 seconds after, how far is she off?

PRACTICAL EXERCISES IN MECHANICS, STATICS, HYDROSTATICS, SOUND, MOTION, GRAVITY, PROJECTILES, AND OTHER BRANCHES OF NATURAL HISTORY.

Question 1. Required the weight of a cast iron ball of 3 inches diameter, supposing the weight of a cubic inch of the metal to be 0·258lb. avoirdupois. Ans. 3·64739lb.

Quest. 2. To determine the weight of a hollow spherical iron shell, 5 inches in diameter, the thickness of the metal being one inch. Ans. 13·78lb.

Quest. 3. Being one day ordered to observe how far a battery of cannon was from me, I counted, by my watch, 17 seconds between the time of seeing the flash and hearing the report; what then was the distance? Ans. $3\frac{5}{9}$ miles.

QUEST. 4. It is proposed to determine the proportional quantities of matter in the earth and moon; the density of the former being to that of the latter, as 10 to 7, and their diameters as 7930 to 2160. Ans. as 71 to 1 nearly.

QUEST. 5. What difference is there, in point of weight, between a block of marble, containing 1 cubic foot and a half, and another of brass of the same dimensions?

Ans. 496lb. 14oz.

QUEST. 6. In the walls of Balbeck in Turkey, the ancient Heliopolis, there are three stones laid end to end, now in sight, that measure in length 61 yards; one of which in particular is 21 yards or 63 feet long, 12 feet thick, and 12 feet broad: now if this block be marble, what power would balance it, so as to prepare it for moving?

Ans. $683\frac{7}{16}$ tons, the burden of an East-India ship.

QUEST. 7. The battering-ram of Vespasian weighed, suppose 10,000 pounds; and was moved, let us admit, with such a velocity, by strength of hand, as to pass through 20 feet in one second of time; and this was found sufficient to demolish the walls of Jerusalem. The question is, with what velocity a 32lb. ball must move, to do the same execution?

Ans. 6250 feet.

QUEST. 8. There are two bodies, of which the one contains 25 times the matter of the other, or is 25 times heavier; but the less moves with 1000 times the velocity of the greater: in what proportion then are the momenta, or forces, with which they move?

Ans. the less moves with a force 40 times greater.

QUEST. 9 A body, weighing 20lb. is impelled by such a force, as to send it through 100 feet in a second; with what velocity then would a body of 8lb. weight move, if it were impelled by the same force? Ans. 250 feet per second.

QUEST. 10. There are two bodies, the one of which weighs 100lb. the other 60; but the less body is impelled by a force 8 times greater than the other; the proportion of the velocities, with which these bodies move, is required?

Ans. the velocity of the greater to that of the less, as 3 to 40.

QUEST. 11. There are two bodies, the greater contains 8 times the quantity of matter in the less, and is moved with a force 48 times greater: the ratio of the velocities of these two bodies is required?

Ans. the greater is to the less, as 6 to 1.

QUEST. 12. There are two bodies, one of which moves 40 times swifter than the other; but the swifter body has

moved only one minute, whereas the other has been in motion 2 hours: the ratio of the spaces described by these two bodies is required?

Ans. the swifter is to the slower, as 1 to 3.

QUEST. 13. Supposing one body to move 30 times swifter than another, as also the swifter to move 12 minutes, the other only 1: what differerce will there be between the spaces described by them, supposing the last has moved 5 feet?

Ans. 1795 feet.

QUEST. 14 There are two bodies, the one of which has passed over 50 miles, the other only 5; and the first had moved with 5 times the celerity of the second; what is the ratio of the times they have been in describing those spaces?

Ans. as 2 to 1.

QUEST. 15. What weight will a man be able to raise, who presses with the force of a hundred and a half, on the end of an equipoised handspike, 100 inches long, meeting with a convenient prop exactly $7\frac{1}{2}$ inches from the lower end of the machine?

Ans. 2072lb.

QUEST. 16. A weight of $1\frac{1}{2}$lb. laid on the shoulder of a man, is no greater burden to him than its absolute weight, or 24 ounces: what difference will he feel between the said weight applied near his elbow, at 12 inches from the shoulder, and in the palm of his hand, 28 inches from the same; and how much more must his muscles then draw, to support it at right angles, that is, having his arm stretched right out?

Ans. 24lb. avoirdupois.

QUEST. 17. What weight hung on at 70 inches from the centre of motion of a steel-yard, will balance a small gun of $9\frac{1}{2}$ cwt. freely suspended at 2 inches distance from the said centre on the contrary side?

Ans. $30\frac{2}{5}$lb.

QUEST. 18. It is proposed to divide the beam of a steel-yard, or to find the points of division where the weights of 1, 2, 3, 4, &c. lb. on the one side, will just balance a constant weight of 95lb. at the distance of 2 inches on the other side of the fulcrum; the weight of the beam being 10lb. and its whole length 36 inches?

Ans. 30, 15, 10, $7\frac{1}{2}$, 6, 5, $4\frac{2}{7}$, $3\frac{3}{4}$, $3\frac{1}{3}$, 3, $2\frac{5}{11}$, $2\frac{1}{2}$, &c.

QUEST. 19. Two men carrying a burden of 200lb. weight between them, hung on a pole, the ends of which rest on their shoulders; how much of this load is borne by each man, the weight hanging 6 inches from the middle, and the whole length of the pole being 4 feet?

Ans. 125lb. and 75lb.

QUEST. 20. If, in a pair of scales, a body weigh 90lb. in one scale, and only 40lb. in the other; required its true weight, and the proportion of the lengths of the two arms of the balance beam, on each side of the point of suspension?

Ans. the weight 60lb. and the proportion 3 to 2.

QUEST. 21. To find the weight of a beam of timber, or other body, by means of a man's own weight, or any other weight. For instance, a piece of tapering timber, 24 feet long, being laid over a prop, or the edge of another beam, is found to balance itself when the prop is 13 feet from the less end; but removing the prop a foot nearer to the said end, it takes a man's weight of 210lb. standing on the less end, to hold it in equilibrium. Required the weight of the tree? Ans. 2520lb.

QUEST. 22. If AB be a cane or walking-stick, 40 inches long, suspended by a string SD fastened to the middle point D: now a body being hung on at E, 6 inches distance from D, is balanced by a weight of 2lb. hung on at the larger end A; but removing the body to F, one inch nearer to D, the 2lb. weight on the other side is moved to G, within 8 inches of D, before the cane will rest in equilibrio. Required the weight of the body? Ans. 24lb.

QUEST. 23. If AB, BC be two inclined planes, of the lengths of 30 and 40 inches, and moveable about the joint at B: what will be the ratio of two weights P, Q, in equilibrio on the planes, in all positions of them: and what will be the altitude BD of the angle B above the horizontal plane AC, when this is 50 inches long?

Ans. BD = 24; and P to Q as AB to BC, or as 3 to 4.

QUEST. 24. A lever, of 6 feet long, is fixed at right angles in a screw, whose threads are one inch asunder, so that the lever turns just once round in raising or depressing the screw one inch. If then this lever be urged by a weight or force of 50lb. with what force will the screw press?

Ans. $22619\frac{1}{2}$lb.

QUEST. 25. If a man can draw a weight of 150lb. up the side of a perpendicular wall, of 20 feet high; what weight will he be able to raise along a smooth plank of 30 feet long, laid aslope from the top of the wall? Ans. 225lb.

QUEST. 26. If a force of 150lb. be applied on the head of a rectangular wedge, its thickness being 2 inches, and the length of its side 12 inches; what weight will it raise or balance perpendicular to its side? Ans. 900lb.

QUEST. 27. If a round pillar of 30 feet diameter be raised

on a plane, inclined to the horizon in an angle of 75°, or the shaft inclining 15 degrees out of the perpendicular; what length will it bear before it overset?

Ans. $30(2 + \sqrt{3})$ or 111·9615 feet.

QUEST. 28. If the greatest angle at which a bank of natural earth will stand, be 45°; it is proposed to determine what thickness an upright wall of stone must be made throughout, just to support a bank of 12 feet high; the specific gravity of the stone being to that of earth, as 5 to 4.

Ans. $\frac{48}{5}\sqrt{\frac{1}{5}}$, or 4·29325 feet.

QUEST. 29. If the stone wall be made like a wedge, or having its upright section a triangle, tapering to a point at top, but its side next the bank of earth perpendicular to the horizon; what is its thickness at the bottom, so as to support the same bank? Ans. $12\sqrt{\frac{1}{5}}$, or 5·36656 feet.

QUEST. 30. But if the earth will only stand at an angle of 30 degrees to the horizontal line; it is required to determine the thickness of wall in both the preceding cases?

Ans. the breadth of the rectangle $12\sqrt{\frac{1}{5}}$, or 5·36656.
but the base of the triangular bank $12\sqrt{\frac{3}{10}}$, or 6·57267.

QUEST. 31. To find the thickness of an upright rectangular wall, necessary to support a body of water; the water being 10 feet deep, and the wall 12 feet high; also the specific gravity of the wall to that of the water, as 11 to 7.

Ans. 4·204374 feet.

QUEST. 32. To determine the thickness of the wall at the bottom, when the section of it is triangular, and the altitudes as before. Ans. 5·1492865 feet.

QUEST. 33. Supposing the distance of the earth from the sun to be 95 millions of miles; I would know at what distance from him another body must be placed, so as to receive light and heat quadruple to that of the earth.

Ans. at half the distance, or $47\frac{1}{2}$ millions.

QUEST. 34. The distance between the earth and the sun being accounted 95 millions of miles, and between Jupiter and the sun 495 millions; the degree of light and heat received by Jupiter, compared with that of the earth is required?

Ans. $\frac{361}{9801}$, or nearly $\frac{1}{27}$ of the earth's light and heat.

QUEST. 35. A certain body on the surface of the earth weighs a cwt., or 112lb.; the question is whither this body must be carried, that it may weigh only 10lb.?

Ans. either at 3·3466 semi-diameters, or $\frac{5}{56}$ of a semi-diameter, from the centre.

QUEST. 36. If a body weigh 1 pound, or 16 ounces, on the surface of the earth; what will its weight be at 50 miles above it, taking the earth's diameter at 7930 miles?

Ans. 15oz. $9\frac{5}{8}$dr. nearly.

QUEST. 37. Whereabouts, in the line between the earth and moon, is their common centre of gravity; supposing the earth's diameter to be 7930 miles, and the moon's 2160; also the density of the former to that of the latter, as 99 to 68, or as 10 to 7 nearly, and their mean distance 30 of the earth's diameters?

Ans. at $\frac{105}{251}$ parts of a diameter from the earth's centre, or $\frac{41}{502}$ parts of a diameter, or 648 miles below the surface.

QUEST. 38. Whereabouts, between the earth and moon, are their attractions equal to each other? Or where must another body be placed, so as to remain suspended in equilibrio, not being more attracted to the one than to the other, or having no tendency to fall either way? Their dimensions being as in the last question.

Ans. From the earth's centre $26\frac{9}{11}$ } of the earth's
From the moon's centre $3\frac{2}{11}$ } diameters.

QUEST. 39. Suppose a stone dropped into an abyss, should be stopped at the end of the 11th second after its delivery; what space would it have gone through? Ans. $1946\frac{1}{12}$ feet.

QUEST. 40. If a heavy body be observed to fall through 100 feet in the last second of time, from what height did it fall, and how long was it in motion?

Ans. time $3\frac{235}{386}$ sec. and height $209\frac{4273}{9264}$ feet.

QUEST. 41. A stone being let fall into a well, it was observed that, after being dropped, it was ten seconds before the sound of the fall at the bottom reached the ear. What is the depth of the well? Ans. 1270 feet nearly.

QUEST. 42. It is proposed to determine the length of a pedulum vibrating seconds, in the latitude of London, where a heavy body falls through $16\frac{1}{12}$ feet in the first second of time? Ans. 39·11 inches.

By experiment this length is found to be $39\frac{1}{8}$ inches.

QUEST. 43. What is the length of a pendulum vibrating in 2 seconds; also in half a second, and in a quarter second?

Ans. the 2 second pendulum $156\frac{1}{2}$
the $\frac{1}{2}$ second pendulum $9\frac{25}{32}$
the $\frac{1}{4}$ second pendulum $2\frac{57}{128}$ inches.

QUEST. 44. What difference will there be in the number of vibrations, made by a pendulum of 6 inches long, and another of 12 inches long, in an hour's time? Ans. $2692\frac{1}{2}$.

QUEST. 45. Observed that while a stone was descending, to measure the depth of a well, a string and plummet, that from the point of suspension, or the place where it was held, to the centre of oscillation, measured just 18 inches, had made 8 vibrations, when the sound from the bottom returned. What was the depth of the well?

Ans. 412·61 feet.

QUEST. 46. If a ball vibrate in the arch of a circle, 10 degrees on each side of the perpendicular; or a ball roll down the lowest 10 degrees of the arch; required the velocity at the lowest point? the radius of the circle, or length of the pendulum, being 20 feet. Ans. 4·4213 feet per second.

QUEST. 47. If a ball descend down a smooth inclined plane, whose length is 100 feet, and altitude 10 feet; how long will it be in descending, and what will be the last velocity?

Ans. the veloc. 25·364 feet per sec. and time 7 8852 sec.

QUEST. 48. If a cannon ball, of 1lb. weight, be fired against a pendulous block of wood, and, striking the centre of oscillation, cause it to vibrate an arc whose cord is 30 inches; the radius of that arc, or distance from the axis to the lowest point of the pendulum, being 118 inches, and the pendulum vibrating in small arcs 40 oscillations per minute. Required the velocity of the ball, and the velocity of the centre of oscillation of the pendulum, at the lowest point of the arc; the whole weight of the pendulum being 500lb.

Ans. veloc. ball 1956·6054 feet per sec.
and veloc. cent. oscil. 3 9054 feet per sec.

QUEST. 49. How deep will a cube of oak sink in common water; each side of the cube being 1 foot, Spec. grav. = 925?

Ans. $11\frac{1}{10}$ inches.

QUEST. 50. How deep will a globe of oak sink in water; the diameter being 1 foot? Ans. 9·9867 inches.

QUEST. 51. If a cube of wood, floating in common water, have three inches of it dry above the water, and $4\frac{8}{107}$ inches dry when in sea-water; it is proposed to determine the magnitude of the cube, and what sort of wood it is made of?

Ans. the wood is oak, and each side 40 inches.

QUEST. 52. An irregular piece of lead ore weighs, in air

12 ounces, but in water only 7; and another fragment weighs in air $14\frac{1}{2}$ ounces, but in water only 9; required their comparative densities, or specific gravities?

Ans. as 145 to 132.

Quest. 53. An irregular fragment of glass, in the scale, weighs 171 grains, and another of magnet 102 grains; but in water the first fetches up no more than 120 grains, and the other 79: what then will their specific gravities turn out to be? Ans. glass to magnet as 3933 to 5202 or nearly as 10 to 13.

Quest. 54. Hiero, king of Sicily, ordered his jeweller to make him a crown, containing 63 ounces of gold. The workmen thought that substituting part silver was only a proper perquisite; which taking air, Archimedes was appointed to examine it; who, on putting it into a vessel of water, found it raised the fluid 8·2245 cubic inches: and having discovered that the inch of gold more critically weighed 10·36 ounces, and that of silver but 5·85 ounces, he found by calculation what part of the king's gold had been changed. And you are desired to repeat the process.

Ans. 28·8 ounces.

Quest. 55. Supposing the cubic inch of common glass weigh 1·4921 ounces troy, the same of sea-water ·59542, and of brandy ·5368; then a seaman having a gallon of this liquor in a glass bottle, which weighs 3·84lb. out of water, and, to conceal it from the officers of the customs, throws it overboard. It is proposed to determine, if it will sink, how much force will just buoy it up? Ans. 14·1496 ounces.

Quest. 56. Another person has half an anker of brandy, of the same specific gravity as in the last question; the wood of the cask suppose measures $\frac{1}{8}$ of a cubic foot; it is proposed to assign what quantity of lead is just requisite to keep the cask and liquor under water? Ans. 89·743 ounces.

Quest. 57. Suppose, by measurement, it be found that a man-of-war, with its ordnance, rigging, and appointments, sinks so deep as to displace 50000 cubic feet of fresh water; what is the whole weight of the vessel? Ans. $1395\frac{1}{10}$ tons.

Quest. 58. It is required to determine what would be the height of the atmosphere, if it were every where of the same density as at the surface of the earth, when the quicksilver in the barometer stands at 30 inches; and also, what would be the height of a water barometer at the same time?

Ans. height of the air $28636\frac{4}{11}$ feet, or 5·4235 miles, height of water 35 feet.

QUEST. 59. With what velocity would each of those three fluids, viz. quicksilver, water, and air, issue through a small orifice in the bottom of vessels, of the respective heights of 30 inches, 35 feet, and 5·5240 miles, estimating the pressure by the whole altitudes, and the air rushing into a vacuum?

Ans. the veloc. of quicksilver 12·681 feet.
the veloc. of water - 47·447
the veloc. of air - 1369·8

QUEST. 60. A very large vessel of 10 feet high (no matter what shape) being kept constantly full of water, by a large supplying cock at the top; if 9 small circular holes, each $\frac{1}{5}$ of an inch diameter, be opened in its perpendicular side at every foot of the depth: it is required to determine the several distances to which they will spout on the horizontal plane of the base, and the quantity of water discharged by all of them in 10 minutes?

Ans. the distances are
$\sqrt{36}$ or 6·00000
$\sqrt{64}$ - 8·00000
$\sqrt{84}$ - 9·16515
$\sqrt{96}$ - 9·79796
$\sqrt{100}$ - 10·00000
$\sqrt{96}$ - 9·79796
$\sqrt{84}$ - 9·16515
$\sqrt{64}$ - 8·00000
$\sqrt{36}$ - 6·00000
and the quantity discharged in 10 min. 123·8849 gallons.

Note. In this solution, the velocity of the water is supposed to be equal to that which is acquired by a heavy body in falling through the whole height of the water above the orifice, and that it is the same in every part of the holes.

QUEST. 61. If the inner axis of a hollow globe of copper, exhausted of air, be 100 feet; what thickness must it be of, that it may just float in the air?

Ans. ·02688 of an inch thick.

QUEST 62. If a spherical balloon of copper, of $\frac{1}{100}$ of an inch thick, have its cavity of 100 feet diameter, and be filled with inflammable air, of $\frac{1}{10}$ of the gravity of common air, what weight will just balance it, and prevent it from rising up into the atmosphere? Ans. 21273lb.

QUEST. 63. If a glass tube, 36 inches long, close at top, be sunk perpendicularly into water, till its lower or open

end be 30 inches below the surface of the water; how high will the water rise within the tube, the quicksilver in the common barometer at the same time standing at $29\frac{1}{2}$ inches?

Ans. 2·26545 inches.

QUEST. 64. If a diving bell, of the form of a parabolic conoid, be let down into the sea to the several depths of 5, 10, 15, and 20 fathoms; it is required to assign the respective heights to which the water will rise within it: its axis and the diameter of its base being each 8 feet, and the quicksilver in the barometer standing at 30·9 inches?

Ans. at	5 fathoms deep the water rises			2·03546 feet.
at 10	-	-	-	3·06393
at 15	-	-	-	3·70267
at 20	-	-	-	4·14658

THE DOCTRINE OF FLUXIONS.

DEFINITIONS AND PRINCIPLES.

Art. 1. IN the Doctrine of Fluxions, magnitudes or quantities of all kinds are considered, not as made up of a number of small parts, but as generated by continued motion, by means of which they increase or decrease. As, a line by the motion of a point; a surface by the motion of a line; and a solid by the motion of a surface. So likewise, time may be considered as represented by a line, increasing uniformly by the motion of a point. And quantities of all kinds whatever, which are capable of increase and decrease, may in like manner be represented by geometrical magnitudes, conceived to be generated by motion. Indeed, notwithstanding all that has been advanced to the contrary, this seems the most natural, as well as the simplest, way of conducting the higher investigations; since it is impossible to conceive a geometrical magnitude to be brought into existence, or to change its magnitude, figure, or place, without motion.

2. Any quantity thus generated, and variable, is called a Fluent, or a Flowing Quantity. And the rate or proportion according to which any flowing quantity increases, at any position or instant, is the Fluxion of the said quantity, at that position or instant: and it is proportional to the magnitude by which the flowing quantity would be uniformly increased

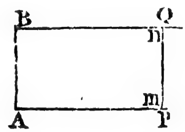
in a given time, with the generating celerity uniformly continued during that time.

3. The small quantities that are actually generated, produced, or described, in any small given time, and by any continued motion, either uniform or variable, are called Increments.

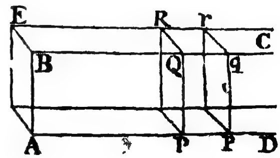
4. Hence, if the motion of increase be uniform, by which increments are generated, the increments will in that case be proportional, or equal, to the measures of the fluxions: but if the motion of increase be accelerated, the increment so generated, in a given finite time, will exceed the fluxion: and if it be a decreasing motion, the increment, so generated, will be less than the fluxion. But if the time be indefinitely small, so that the motion be considered as uniform for that instant; then these nascent increments will always be proportional, or equal, to the fluxions, and may be substituted instead of them, in any calculation.

5. To illustrate these definitions: Suppose a point m be conceived to move from the position A, and to generate a line AP, by a motion any how regulated; and suppose the celerity of the point m, at any position P, to be such, as would, if from thence it should become or continue uniform, be sufficient to cause the point to describe, or pass uniformly over, the distance Pp, in the given time allowed for the fluxion: then will the said line Pp represent the fluxion of the fluent, or flowing line, AP, at that position.

6. Again, suppose the right line mn to move, from the position AB, continually parallel to itself, with any continued motion, so as to generate the fluent or flowing rectangle ABQP, while the point m describes the line AP: also, let the distance Pp be taken, as before, to express the fluxion of the line or base AP; and complete the rectangle PQqp. Then, like as Pp is the fluxion of the line AP, so is Pq the fluxion of the flowing parallelogram AQ; both these fluxions, or increments, being uniformly described in the same time.

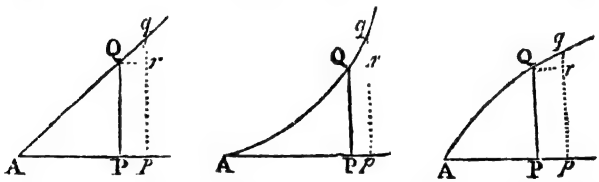


7. In like manner, if the solid AERP be conceived to be generated by the plane PQR, moving from the position ABE, always parallel to itself, along the line AD; and if Pp denote the fluxion of the line AP: Then, like as the



rectangle PQqp, or PQ $\times$ Pp, denotes the fluxion of the flowing rectangle ABQP, so also shall the fluxion of the variable solid, or prism ABERQP, be denoted by the prism PQRrqp, or the plane PR $\times$ Pp. And, in both these last two cases, it appears that the fluxion of the generated rectangle, or prism, is equal to the product of the generating line, or plane, drawn into the fluxion of the line along which it moves.

8. Hitherto the generating line, or plane, has been considered as of a constant and invariable magnitude; in which case the fluent, or quantity generated, is a rectangle, or a prism, the former being described by the motion of a line, and the latter by the motion of a plane. So, in like manner are other figures, whether plane or solid, conceived to be described by the motion of a Variable Magnitude, whether it be a line or a plane. Thus, let a variable line PQ be carried by a parallel motion along AP; or while a point P is carried along, and describes the line AP, suppose another point



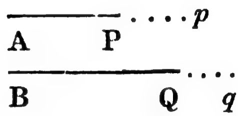
Q to be carried by a motion perpendicular to the former, and to describe the line PQ: let pq be another position of PQ, indefinitely near to the former; and draw Qr parallel to AP. Now in this case there are several fluents, or flowing quantities, with their respective fluxions; namely, the line or fluent AP, the fluxion of which is Pp or Qr; the line or fluent PQ, the fluxion of which is rq; the curve or oblique line AQ, described by the oblique motion of the point Q, the fluxion of which is Qq; and lastly, the surface APQ, described by the variable line PQ, the fluxion of which is the rectangle PQrp, or PQ $\times$ Pp. In the same manner may any solid be conceived to be described, by the motion of a variable plane parallel to itself, substituting the variable plane for the variable line; in which case the fluxion of the solid, at any position, is represented by the variable plane, at that position, drawn into the fluxion of the line along which it is carried.

9. Hence then it follows in general, that the fluxion of any figure, whether plane or solid, at any position, is equal to the section of it, at that position, drawn into the fluxion of the axis, or line along which the variable section is sup-

posed to be perpendicularly carried; that is, the fluxion of the figure AQP, is equal to the plane PQ $\times$ Pp, when that figure is a solid, or to the ordinate PQ $\times$ Pp, when the figure is a surface.

10. It also follows from the same premises, that in any curve, or oblique line AQ, whose absciss is AP, and ordinate is PQ, the fluxions of these three form a small right-angled plane triangle Qqr; for Qr = Pp is the fluxion of the absciss AP, qr the fluxion of the ordinate PQ, and Qq the fluxion of the curve or right line AQ. And consequently that, in any curve, the square of the fluxion of the curve, is equal to the sum of the squares of the fluxions of the absciss and ordinate, when these two are at right angles to each other.

11. From the premises it also appears, that contemporaneous fluents, or quantities that flow or increase together, which are always in a constant ratio to each other, have their fluxions also in the same constant ratio, at every position. For, let AP and BQ be two contemporaneous fluents, described in the same time by the motion of the points P and Q, the contemporaneous positions being P, Q, and p, q; and let AP be to BQ, or Ap to Bq, constantly in the ratio of 1 to n.



Then - - - is $n \times$ AP = BQ,
and $n \times$ Ap = Bq;
therefore, by subtraction, $n \times$ Pp = Qq;
that is, the fluxion - Pp : fluxion Qq :: 1 : n,
the same as the fluent AP : fluent BQ :: 1 : n,
or, the fluxions and fluents are in the same constant ratio.

But if the ratio of the fluents be variable, so will that of the fluxions be also, though not in the same variable ratio with the former, at every position.

NOTATION, &c.

12. To apply the foregoing principles to the determination of the fluxions of algebraic quantities, by means of which those of all other kinds are assigned, it will be necessary first to premise the notation commonly used in this science, with some observations. As, first, that the final letters of the alphabet z, y, x, u, &c. are used to denote variable or flowing quantities; and the initial letters a, b, c, d, &c. to denote constant or invariable ones: Thus, the variable base AP of the flowing rectangular figure ABQP, in art. 6, may be repre-

sented by x; and the invariable altitude PQ, by a: also, the variable base or absciss AP, of the figures in art. 8, may be represented by x, the variable ordinate PQ, by y; and the variable curve or line AQ, by z.

Secondly, that the fluxion of a quantity denoted by a single letter, is usually represented by the same letter with a point over it: Thus, the fluxion of x is expressed by $\dot{x}$, the fluxion of y by $\dot{y}$, and the fluxion of z by $\dot{z}$. Sometimes, however, the fluxion of a variable quantity, especially if it be a compound one, is denoted by the Greek character φ before it. Thus, the fluxion of xy may be denoted by $\varphi(xy)$; the fluxion of xyz by $\varphi(xyz)$. As to the fluxions of constant or invariable quantities, as of a, b, c, &c. they are equal to nothing, because they do not flow or change their magnitude.

Thirdly, that the increments of variable or flowing quantities, are also denoted by the same letters with a small ′ over them: Thus, the increments of x, y, z, are x', y', z'.

13. From these notations, and the foregoing principles, the quantities, and their fluxions, there considered, will be denoted as below. Thus, in all the foregoing figures, put

the variable or flowing line - - AP $= x$,
in art. 6, the constant line - - PQ $= a$,
in art. 8, the variable ordinate - PQ $= y$,
also, the variable line or curve - AQ $= z$:

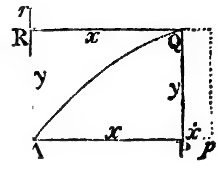
Then shall the several fluxions be thus represented, namely,

$\dot{x} =$ Pp the fluxion of the line AP,
$a\dot{x} =$ PQqp the fluxion of ABQP in art. 6,
$y\dot{x} =$ PQrp the fluxion of APQ in art. 8,
$\dot{z} =$ Qq $= \sqrt{(\dot{x}^2 + \dot{y}^2)}$ the fluxion of AQ; and
$a\dot{x} =$ Pr the fluxion of the solid in art. 7, if a denote the constant generating plane PQR; also
$nx =$ BQ in the figure to art. 11, and
$n\dot{x} =$ Qq the fluxion of the same.

14. The principles and notation being now laid down, we may proceed to the practice and rules of this doctrine; which consist of two principal parts, called the Direct and Inverse Method of Fluxions; namely, the direct method, which consists in finding the fluxion of any proposed fluent or flowing quantity; and the inverse method, which consists in finding the fluent of any proposed fluxion. As to the former of these two problems, it can always be determined, and that in finite algebraic terms; but the latter, or finding of fluents, can only be effected in some certain cases, except by means of infinite series.—First then, of

THE DIRECT METHOD OF FLUXIONS.

To find the Fluxion of the Product or Rectangle of two Variable Quantities.



15. Let ARQP, $= xy$, be the flowing or variable rectangle, generated by two lines PQ and RQ, moving always perpendicular to each other, from the positions AR and AP; denoting the one by x, and the other by y; supposing x and y to be so related, that the curve line AQ may always pass through the intersection Q of those lines, or the opposite angle of the rectangle.

Now, the rectangle consists of the two trilinear spaces APQ, ARQ, of which, the

fluxion of the former is PQ $\times$ Pp, or $y\dot{x}$,

that of the latter is - RQ $\times$ Rr, or $x\dot{y}$, by art. 8;

therefore the sum of the two $\dot{x}y + x\dot{y}$, is the fluxion of the whole rectangle xy or ARQP.

The Same Otherwise.

16. Let the sides of the rectangle x and y, by flowing, become $x + x'$ and $y + y'$: then the product of these two, or $xy + xy' + yx' + x'y'$ will be the new or contemporaneous value of the flowing rectangle PR or xy: subtract the one value from the other, and the remainder, $xy' + yx' + x'y'$, will be the increment generated in the same time as x' or y'; of which the last term $x'y'$ is nothing, or indefinitely small, in respect of the other two terms, because x' and y' are indefinitely small in respect of x and y; which term being therefore omitted, there remains $xy' + yx'$ for the value of the increment; and hence, by substituting $\dot{x}$ and $\dot{y}$ for x' and y', to which they are proportional, there arises $x\dot{y} + y\dot{x}$ for the true value of the fluxion of xy; the same as before.

17. Hence may be easily derived the fluxion of the powers and products of any number of flowing or variable quantities whatever; as of xyz, or $uxyz$, or $vuxyz$, &c. And first, for the fluxion of xyz: put $p = xy$, and the whole given fluent $xyz = q$, or $q = xyz = pz$. Then, taking the fluxions of $q = pz$, by the last article, they are $\dot{q} = \dot{p}z + p\dot{z}$; but $p = xy$, and so $\dot{p} = \dot{x}y + x\dot{y}$ by the same article; substituting therefore these values of p and $\dot{p}$ instead of them, in the value of $\dot{q}$, this becomes $\dot{q} = \dot{x}yz + x\dot{y}z + xy\dot{z}$,

the fluxion of xyz required; which is therefore equal to the sum of the products, arising from the fluxion of each letter, or quantity, multiplied by the product of the other two.

Again, to determine the fluxion of $uxyz$, the continual product of four variable quantities; put this product, namely $uxyz$, or $qu = r$, where $q = xyz$ as above. Then, taking the fluxions by the last article, $\dot{r} = \dot{q}u + q\dot{u}$; which, by substituting for q and $\dot{q}$ their values as above, becomes - - $\dot{r} = \dot{u}xyz + u\dot{x}yz + ux\dot{y}z + uxy\dot{z}$, the fluxion of $uxyz$ as required: consisting of the fluxion of each quantity, drawn into the products of the other three.

In the very same manner it is found, that the fluxion of $vuxyz$ is $\dot{v}uxyz + v\dot{u}xyz + vu\dot{x}yz + vux\dot{y}z + vuxy\dot{z}$; and so on, for any number of quantities whatever; in which it is always found, that there are as many terms as there are variable quantities in the proposed fluent; and that these terms consist of the fluxion of each variable quantity, multiplied by the product of all the rest of the quantities.

18. Hence is easily derived the fluxion of any power of a variable quantity, as of x^2, or x^3, or x^4, &c. For, in the product or rectangle xy, if $x = y$, then is $xy = xx$ or x^2, and also its fluxion $\dot{x}y + x\dot{y} = \dot{x}x + x\dot{x}$ or $2x\dot{x}$, the fluxion of x^2.

Again, if all the three x, y, z be equal; then is the product of the three $xyz = x^3$; and consequently its fluxion $\dot{x}yz + x\dot{y}z + xy\dot{z} = \dot{x}xx + x\dot{x}x + xx\dot{x}$ or $3x^2\dot{x}$, the fluxion of x^3.

In the same manner, it will appear that

the fluxion of x^4 is $= 4x^3\dot{x}$, and
the fluxion of x^5 is $= 5x^4\dot{x}$, and, in general,
the fluxion of x^n is $= nx^{n-1}\dot{x}$;

where n is any positive whole number whatever.

That is, the fluxion of any positive integral power, is equal to the fluxion of the root ($\dot{x}$), multiplied by the exponent of the power (n), and by the power of the same root whose index is less by 1, (x^{n-1})*.

* In the text, the fluxion of the product of two, three, or more, variable quantities is found, and thence, by supposing them to become equal the fluxions of the square, cube, &c. of a variable quantity, are inferred. Sometimes, the investigation commences with the fluxion of a square, and proceeds thence to that of a rectangle.

Let $x—s$ and x be two states of the same line generated by an equable motion; then, while the line $x—s$ by flowing equably becomes x, its square $(x - s)^2$ will become x^2. That is, while

And thus, the fluxion of $a + cx$ being $c\dot{x}$,
that of $(a + cx)^2$ is $2c\dot{x} \times (a + cx)$ or $2ac\dot{x} + 2c^2x\dot{x}$,
that of $(a + cx^2)^2$ is $4cx\dot{x} \times (a + cx^2)$ or $4acx\dot{x} + 4c^2x^3\dot{x}$,
that of $(x^2 + y^2)^2$ is $(4x\dot{x} + 4y\dot{y}) \times (x^2 + y^2)$,
that of $(x + cy^2)^3$ is $(3\dot{x} + 6cy\dot{y}) \times (x + cy^2)^2$.

19. From the conclusions in the same article, we may also derive the fluxion of any fraction, or the quotient of one variable quantity divided by another, as of

$\frac{x}{y}$. For, put the quotient or fraction $\frac{x}{y} = q$; then, multiplying by the denominator, $x = qy$; and, taking the fluxions,

$$\dot{x} = \dot{q}y + q\dot{y}, \text{ or } \dot{q}y = \dot{x} - q\dot{y}; \text{ and, by division,}$$

$$\dot{q} = \frac{\dot{x}}{y} - \frac{q\dot{y}}{y} = \text{(by substituting the value of } q, \text{ or } \frac{x}{y}\text{)},$$

$$\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2} = \frac{\dot{x}y - x\dot{y}}{y^2}, \text{ the fluxion of } \frac{x}{y}, \text{ as required.}$$

That is, the fluxion of any fraction, is equal to the fluxion of the numerator drawn into the denominator, minus the fluxion of the denominator drawn into the numerator, and the remainder divided by the square of the denominator.

the space s is described equably by the flowing line, the space $x^2 - (x-s)^2 = 2sx - s^2$ will be described by the flowing square of that line, and this latter is the space which *would* have been generated in the same time by a certain magnitude (whether assignable or not) moving uniformly. Hence, the fluxion of the flowing magnitude $(x-s)$, is to the fluxion of the flowing magnitude $(x-s)^2$, as s to $2sx - s^2$, or as 1 to $2x - s$: and as this must obtain in all possible values of $x - s$, it must obtain in the ultimate state, when $(x-s)$ by flowing, becomes x; and then, s vanishing, the ratio becomes 1 to $2x$. That is, the ratio of the fluxions of x and x^2 is that of 1 to $2x$. Consequently, if $\dot{x}$ denote the fluxion of x, then will $2x\dot{x}$ denote the fluxion of x^2.

The fluxion of the square of a quantity being thus found, that of any product is easily assigned. Thus, to determine the fluxions of the product of xy:

Put $x + y = s$; then $\varphi(x+y) = \dot{x} + \dot{y} = \dot{s}$,
also, $x^2 + 2xy + y^2 = s^2$; $\therefore 2xy = s^2 - x^2 - y^2$,
and $xy = \frac{1}{2}s^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2$,

$$\begin{aligned} \therefore \text{ by the above } \varphi(xy) &= s\dot{s} - x\dot{x} - y\dot{y}, \\ &= s(\dot{x} + \dot{y}) - x\dot{x} - y\dot{y}, \\ &= (x + y)(\dot{x} + \dot{y}) - x\dot{x} - y\dot{y}, \\ &= x\dot{x} + x\dot{y} + y\dot{x} + y\dot{y} - x\dot{x} - y\dot{y}, \\ &= x\dot{y} + y\dot{x}, \end{aligned}$$

agreeing with the result in art. 15.

So that the fluxion of $\frac{ax}{y}$ is $a \times \frac{\dot{x}y - x\dot{y}}{y^2}$ or $\frac{a\dot{x}y - ax\dot{y}}{y^2}$.

20. Hence too is easily derived the fluxion of any negative integer power of a variable quantity, as of x^{-n}, or $\frac{1}{x^n}$, which is the same thing. For here the numerator of the fraction is 1, whose fluxion is nothing; and therefore, by the last article, the fluxion of such a fraction, or negative power, is barely equal to minus the fluxion of the denominator, divided by the square of the said denominator. That is, the fluxion of x^{-n}, or $\frac{1}{x^n}$ is $-\frac{nx^{n-1}\dot{x}}{x^{2n}}$ or $-\frac{n\dot{x}}{x^{n+1}}$ or $-nx^{-n-1}\dot{x}$; or the fluxion of any negative integer power of a variable quantity, as x^{-n}, is equal to the fluxion of the root, multiplied by the exponent of the power, and by the next power less by 1; the same rule as for positive powers.

The same thing is otherwise obtained thus: Put the proposed fraction, or quotient $\frac{1}{x^n} = q$; then is $qx^n = 1$; and, taking the fluxions, we have
$\dot{q}x^n + qnx^{n-1}\dot{x} = 0$; hence $\dot{q}x^n = -qnx^{n-1}\dot{x}$; divide by x^n, then $\dot{q} = -\frac{qn\dot{x}}{x} =$ (by substituting $\frac{1}{x^n}$ for q), $\frac{-n\dot{x}}{x^{n+1}}$ or $=$ $-nx^{-n-1}\dot{x}$; the same as before.

Hence the fluxion of x^{-1} or $\frac{1}{x}$ is $-x^{-2}\dot{x}$, or $-\frac{\dot{x}}{x^2}$,

that of - x^{-2} or $\frac{1}{x^2}$ is $-2x^{-3}\dot{x}$ or $-\frac{2\dot{x}}{x^3}$,

that of - x^{-3} or $\frac{1}{x^3}$ is $-3x^{-4}\dot{x}$ or $-\frac{3\dot{x}}{x^4}$,

that of - ax^{-4} or $\frac{a}{x^4}$ is $-4ax^{-5}\dot{x}$ or $-\frac{4a\dot{x}}{x^5}$,

that of - x^{-n} or $\frac{1}{x^n}$ is $-\frac{n\dot{x}}{x^{n+1}}$,

that of $(a+x)^{-1}$ or $\frac{1}{a+x}$ is $-(a+x)^{-2}\dot{x}$ or

that of $c(a+3x^2)^{-2}$ or $\frac{c}{(a+3x^2)^2}$ is $-12cx\dot{x} \times (a+3x^2)^{-3}$,

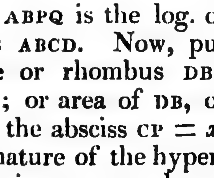
or $-\frac{12cx\dot{x}}{(a+3x^2)^3}$.

21. Much in the same manner is obtained the fluxion of any fractional power of a fluent quantity, as of $x^{\frac{m}{n}}$, or $\sqrt[n]{x^m}$. For, put the proposed quantity $x^{\frac{m}{n}} = q$; then, raising each side to the n power, gives $x^m = q^n$; taking the fluxions, gives $mx^{m-1}\dot{x} = nq^{n-1}\dot{q}$; then dividing by nq^{n-1}, gives $\dot{q} = \frac{mx^{m-1}\dot{x}}{nq^{n-1}} = \frac{m}{n}x^{\frac{m}{n}-1}\dot{x}$.

Which is still the same rule, as before, for finding the fluxion of any power of a fluent quantity, and which therefore is general, whether the exponent be positive or negative, integral or fractional. And hence the fluxion of $ax^{\frac{3}{2}}$ is $\frac{3}{2}ax^{\frac{1}{2}}\dot{x}$; that of $ax^{\frac{1}{2}}$ is $\frac{1}{2}ax^{\frac{1}{2}-1}\dot{x} = \frac{1}{2}ax^{-\frac{1}{2}}\dot{x} = \frac{a\dot{x}}{2x^{\frac{1}{2}}} = \frac{a\dot{x}}{2\sqrt{x}}$; and that of $\sqrt{(a^2-x^2)}$ or $(a^2-x^2)^{\frac{1}{2}}$ is $\frac{1}{2}(a^2-x^2)^{-\frac{1}{2}} \times -2x\dot{x} = \frac{-x\dot{x}}{\sqrt{(a^2-x^2)}}$.

22. Having now found out the fluxions of all the ordinary forms of algebraical quantities; it remains to determine those of logarithmic expressions; and also of exponential ones, that is, such powers as have their exponents variable or flowing quantities. And first, for the fluxion of Napier's, or the hyperbolic logarithm.

23. Now, to determine this from the nature of the hyperbolic spaces. Let A be the principal vertex of an hyperbola, having its asymptotes CD, CP, with the ordinates DA, BA, PQ, &c. parallel to them. Then, from the nature of the hyperbola and of logarithms, it is known, that any space ABPQ is the log. of the ratio of CB to CP, to the modulus ABCD. Now, put $1 =$ CB or BA the side of the square or rhombus DB; $m =$ the modulus, or CB $\times$ BA $\times$ sin. C; or area of DB, or sine of the angle C to the radius 1; also the absciss CP $= x$, and the ordinate PQ $= y$. Then, by the nature of the hyperbola, CP $\times$ PQ is always equal to DB, that is, $xy = m$; hence



$y = \frac{m}{x}$, and the fluxion of the space, $\dot{x}y$ is $\frac{m\dot{x}}{x} =$ PQqp the fluxion of the log. of x, to the modulus m. And, in the hyperbolic logarithms, the modulus m being 1, there-

fore $\frac{\dot{x}}{x}$ is the fluxion of the hyp. log. of x; which is therefore equal to the fluxion of the quantity, divided by the quantity itself.

Hence the fluxion of the hyp. log.

of $1+x$ is $\frac{\dot{x}}{1+x}$,

of $1-x$ is $\frac{-\dot{x}}{1-x}$,

of $x+z$ is $\frac{\dot{x}+\dot{z}}{x+z}$,

of $\frac{a+x}{a-x}$ is $\frac{\dot{x}(a-x)+\dot{x}(a+x)}{(a-x)^2} \times \frac{a-x}{a+x} = \frac{2a\dot{x}}{a^2-x^2}$,

of ax^n is $\frac{nax^{n-1}\dot{x}}{ax^n} = \frac{n\dot{x}}{x}$.

24. By means of the fluxions of logarithms, are usually determined those of exponential quantities, that is, quantities which have their exponent a flowing or variable letter. These exponentials are of two kinds, namely, when the root is a constant quantity, as e^x, and when the root is variable as well as the exponent, as y^x.

25. In the first case, put the exponential, whose fluxion is to be found, equal to a single variable quantity z, namely, $z=e^x$; then take the logarithm of each, so shall log. $z = x \times$ log. e; take the fluxions of these, so shall $\frac{\dot{z}}{z} = \dot{x} \times$ log. e, by the last article; hence $\dot{z} = z\dot{x} \times$ log. $e = e^x\,\dot{x} \times$ log. e, which is the fluxion of the proposed quantity e^x or z; and which therefore is equal to the said given quantity drawn into the fluxion of the exponent, and into the log. of the root.

Hence also, the fluxion of $(a+c)^{nx}$ is $(a+c)^{nx} \times n\dot{x} \times$ log. $(a+c)$.

26. In like manner, in the second case, put the given quantity $y^x = z$; then the logarithms give log. $z = x \times$ log. y, and the fluxions give $\frac{\dot{z}}{z} = \dot{x}$. log. $y + x \,.\, \frac{\dot{y}}{y}$; hence $\dot{z} = z\dot{x}$. log. $y + \frac{zx\dot{y}}{y} =$ (by substituting y^x for z) $y^x\dot{x}$. log. $y + xy^{x-1}\dot{y}$, which is the fluxion of the proposed quantity y^x; and which therefore consists of two terms, of which the one is the fluxion of the given quantity considering the

exponent as constant, and the other the fluxion of the same quantity considering the root as constant.

27. The fluxions of the usual trigonometrical quantities, sin. z, cos. z, &c. are easily found by blending these principles with the analytical formulæ at pa. 18, of this volume. We assume the proportionality of the increments, and of their contemporaneous fluxions, and proceed thus:

To find φ sin. z, we suppose that by a motion of one of the legs including the angle, it becomes $z + x'$ or $z + \dot{z}$. Then φ sin. $z = \sin. (z + \dot{z}) - \sin. z$. But by equa. 9, p. 18, we have

$$\sin. (z + \dot{z}) = \sin. z. \cos. \dot{z} + \sin. \dot{z} \cos. z.$$

But the sine of an arc indefinitely small does not differ sensibly from that arc itself, nor its cosine differ perceptibly from radius; hence we have $\sin. \dot{z} = \dot{z}$, and $\cos. \dot{z} = 1$; and therefore $\sin. (z + \dot{z}) = \sin. z + \dot{z} \cos. z$; whence $\sin. (z + \dot{z}) - \sin. z$, or $\varphi (\sin. z) = \dot{z} \cos. z$, viz. the fluxion of the sine of an arc whose radius is unity, is equal to the product of the fluxion of the arc into the cosine of the same arc.

28. In like manner, the fluxion of cos. z, or $\cos. (z + \dot{z}) - \cos. z = \cos. z \cos. \dot{z} - \sin. z \sin. \dot{z} - \cos. z$, or since $\cos. (z + \dot{z}) = \cos. z \cos. \dot{z} - \sin. z \sin. \dot{z}$; therefore, because $\sin. \dot{z} = \dot{z}$, and $\cos. \dot{z} = 1$, we have $\varphi \cos. z = \cos. z - \dot{z} \sin. z - \cos. z = - \dot{z} \sin. z$, that is, the fluxion of the cosine of an arc, radius being 1, is found by multiplying the fluxion of the arc (taken with a contrary sign) by the sine of the same arc.

29. By means of these two formulæ, many other fluxional expressions may be found, viz.

$$\varphi \cos. mz = - m\dot{z} \sin. mz.$$

$$\varphi \sin. mz = + m\dot{z} \cos. mz.$$

$$\varphi \tan. z = \frac{\dot{z}}{\cos.^2 z} = \dot{z} \sec.^2 z.$$

$$\varphi \text{cotan.} z = - \frac{\dot{z}}{\sin.^2 z} = - \dot{z} \text{cosec.}^2 z.$$

$$\dot{z} \sin. z \qquad \frac{\dot{z} \tan. z}{\cos. z}.$$

$$\varphi \text{cosec.} z = \frac{\dot{z} \cos. z}{\sin.^2 z} - \frac{\dot{z} \cot. z}{\sin. z}$$

$$\varphi \sin.^m z = m \sin.^{m-1} z\, \dot{z} \cos. z.$$

$$\varphi \cos.^m z = - m \cos.^{m-1} z\, \dot{z} \sin. z.$$

30. Hence, by the way, will flow this useful practical conclusion, that if z be any arc, then

$$\dot{z} = \frac{\dot{\phi} \text{ sin. } z}{\text{cos. } z} = \frac{- \dot{\phi} \text{ cos. } z}{\text{sin. } z} = \text{cos.}^2 z \ \dot{\phi} \text{ tan. } z.$$

$$= \frac{\dot{\phi} \text{ tan. } z}{1 + \text{tan.}^2 z} = - \dot{\phi} \text{ cot. } z \text{ sin.}^2 z = \frac{- \dot{\phi} \text{ cot. } z}{1 + \text{cot.}^2 z}.$$

OF SECOND, THIRD, &c. FLUXIONS.

Having explained the manner of considering and determining the first fluxions of flowing or variable quantities; it remains now to consider those of the higher orders, as second, third, fourth, &c. fluxions.

31. If the rate or celerity with which any flowing quantity changes its magnitude be constant, or the same at every position; then is the fluxion of it also constantly the same. But if the variation of magnitude be continually changing, either increasing or decreasing; then will there be a certain degree of fluxion peculiar to every point or position; and the rate of variation or change in the fluxion, is called the Fluxion of the Fluxion, or the Second Fluxion of the given fluent quantity. In like manner, the variation or fluxion of this second fluxion, is called the Third Fluxion of the first proposed fluent quantity; and so on.

These orders of fluxions are denoted by the same fluent letter with the corresponding number of points over it: namely, two points for the second fluxion, three points for the third fluxion, four points for the fourth fluxion, and so on. So, the different orders of the fluxion of x, are $\dot{x}$, $\ddot{x}$, $\dddot{x}$, $\ddddot{x}$, &c.; where each is the fluxion of the one next before it.

32. This description of the higher orders of fluxions may be illustrated by the figures exhibited in art. 8, where, if x denote the absciss AP, and y the ordinate PQ; and if the ordinate PQ or y flow along the absciss AP or x, with a uniform motion; then the fluxion of x, namely, $\dot{x} = \text{P}p$ or Qr, is a constant quantity, or $\ddot{x} = 0$, in all the figures. Also, in fig. 1, in which AQ is a right line, $\dot{y} = rq$, or the fluxion of PQ, is a constant quantity, or $\ddot{y} = 0$; for the angle Q, = the angle A, being constant, Qr is to rq, or $\dot{x}$ to $\dot{y}$, in a constant ratio. But in the 2d fig. rq, or the fluxion of PQ, continually increases more and more; and in fig. 3 it continually decreases more and more, and therefore in both these cases y

has a second fluxion, being positive in fig. 2, but negative in fig. 3. And so on, for the other orders of fluxions.

Thus if, for instance, the nature of the curve be such, that x^3 is every where equal to a^2y; then, taking the fluxions, it is $a^2\dot{y} = 3x^2\dot{x}$; and, considering $\dot{x}$ always as a constant quantity, and taking always the fluxions, the equations of the several orders of fluxions will be as below, viz.

the 1st fluxions $a^2\dot{y} = 3x^2\dot{x}$,

the 2d fluxions $a^2\ddot{y} = 6x\dot{x}^2$,

the 3d fluxions $a^2\dddot{y} = 6\dot{x}^3$,

the 4th fluxions $a^2\ddddot{y} = 0$,

and all the higher fluxions also $= 0$, or nothing.

Also the higher orders of fluxions are found in the same manner as the lower ones. Thus,

the first fluxion of y^3 is - - $3y^2\dot{y}$;

its 2d flux. or the flux. of $3y^2\dot{y}$, considered as the rectangle of $3y^2$, and $\dot{y}$, is - - - - $\Big\}\ 3y^2\ddot{y} + 6y\dot{y}^2$:

and the flux. of this again, or the 3d flux of y^3, is - - - $\Big\}\ 3y^2\dddot{y} + 18y\dot{y}\ddot{y} + 6\dot{y}^3$.

33. If the function proposed were ax^n, we should find $\varphi\ ax^n = nax^{n-1}\dot{x}$; the factors n, a, and $\dot{x}$ being regarded as constant in the first fluxion $nax^{n-1}\dot{x}$, to obtain the second fluxion it will suffice to make x^{n-1} flow, and to multiply the result by $na\dot{x}$; but $\varphi\ x^{n-1} = (n-1)\ x^{n-2}\dot{x}$; we have, therefore,

2nd $\varphi\ ax^n = n\ (n-1)\ ax^{n-2}\ \dot{x}^2$.

3rd $\varphi\ ax^n = n\ (n-1)\ (n-2)\ ax^{n-3}\ \dot{x}^3$.

4th $\varphi\ ax^n = n\ (n-1)\ (n-2)\ (n-3)\ ax^{n-4}\ \dot{x}^4$.

&c. $=$ &c.

mth $\varphi\ ax^n = n\ (n-1)\ (n-2) \ldots . (n-m+1)$
$a^{n-m}\ \dot{x}^m$,

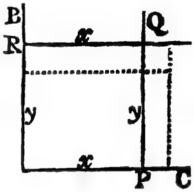
m being supposed not to exceed n, for it is manifest that in the case of n being integral, the function ax^n has only a limited number of fluxions, of which the most elevated is the nth, and which of course is expressed by the formula,

nth $\varphi\ ax^n = n\ (n-1)\ (n-2) \ldots . 3 . 2 . 1 . a\dot{x}^n$

in which state it admits no longer of being put into fluxions, as it contains no variable quantity, or, in other words, its fluxion is equal to zero.

34. In the foregoing articles, it has been supposed that

the fluents increase, or that their fluxions are positive; but it often happens that some fluents decrease, and that therefore their fluxions are negative: and whenever this is the case, the sign of the fluxion must be changed, or made contrary to that of the fluent. So, of the rectangle xy, when both x and y increase together, the fluxion is $\dot{x}y + x\dot{y}$: but if one of them, as y, decrease, while the other, x, increases; then, the fluxion of y being $-\dot{y}$, the fluxion of xy will in that case be $\dot{x}y - x\dot{y}$. This may be illustrated by the annexed rectangle, APQR $= xy$, supposed to be generated by the motion of the line PQ from A towards C, and by the motion of the line RQ from B towards A: For, by the motion of PQ, from A towards C, the rectangle is increased, and its fluxion is $+ \dot{x}y$; but, by the motion of RQ, from B towards A, the rectangle is decreased, and the fluxion of the decrease is $x\dot{y}$; therefore, taking the fluxion of the decrease from that of the increase, the fluxion of the rectangle xy, when x increases and y decreases, is $\dot{x}y - x\dot{y}$.



35. We may now collect the principal rules, which have been demonstrated in the foregoing articles, for finding the fluxions of all sorts of quantities. And hence,

1st, *For the fluxion of any Power of a flowing quantity.*—Multiply all together the exponent of the power, the fluxion of the root, and the power next less by 1 of the same root.

2d, *For the fluxion of the Rectangle of two quantities.*—Multiply each quantity by the fluxion of the other, and connect the two products together by their proper signs.

3d, *For the fluxion of the Continual Product of any number of flowing quantities.*—Multiply the fluxion of each quantity by the product of all the other quantities, and connect all the products together by their proper signs.

4th, *For the fluxion of a Fraction.*—From the fluxion of the numerator drawn into the denominator, subtract the fluxion of the denominator drawn into the numerator, and divide the result by the square of the denominator.

5th, *Or, the 2d, 3d, and 4th cases may be all included under one, and performed thus.*—Take the fluxion of the given expression as often as there are variable quantities in it, supposing first only one of them variable, and the rest

constant; then another variable, and the rest constant; and so on, till they have all in their turns been singly supposed variable; and connect all these fluxions together with their own signs.

6th, *For the Fluxion of a Logarithm.*—Divide the fluxion of the quantity by the quantity itself, and multiply the result by the modulus of the system of logarithms.

Note.—The modulus of the hyperbolic logarithms is 1, and the modulus of the common logs, is 0.43429448, &c.

7th, *For the fluxion of an Exponential quantity, having the Root Constant.*—Multiply all together, the given quantity, the fluxion of its exponent, and the hyp. log. of the root.

8th, *For the fluxion of an Exponential quantity, having the Root Variable.*—To the fluxion of the given quantity, found by the 1st rule, as if the root only were variable, add the fluxion of the same quantity found by the 7th rule, as if the exponent only were variable; and the sum will be the fluxion for both of them variable.

Note.—When the given quantity consists of several terms, find the fluxion of each term separately, and connect them all together with their proper signs; also, for the fluxions of trigonometrical formulæ, take the formulæ in arts. 27—30.

36. PRACTICAL EXAMPLES TO EXERCISE THE FOREGOING RULES.

1. The fluxion of axy is
2. The fluxion of $bxyz$ is
3. The fluxion of $cx \times (ax - cy)$ is
4. The fluxion of $x^m y^n$ is
5. The fluxion of $x^m y^n z^r$ is
6. The fluxion of $(x + y) \times (x - y)$ is
7. The fluxion of $2ax^2$ is
8. The fluxion of $2x^3$ is
9. The fluxion of $3x^4 y$ is
10. The fluxion of $4x^{\frac{2}{3}} y^4$ is
11. The fluxion of $ax^2 y - x^{\frac{1}{2}} y^3$ is
12. The fluxion of $4x^4 - x^2 y + 2byz$ is
13. The fluxion of $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$ is

14. The fluxion of $\sqrt[n]{x^m}$ or $x^{\frac{m}{n}}$ is

15. The fluxion of $\dfrac{1}{\sqrt[n]{x^m}}$ or $\dfrac{1}{x^{\frac{m}{n}}}$ or $x^{-\frac{m}{n}}$ is

16. The fluxion of $\sqrt{x}$ or $x^{\frac{1}{2}}$ is

17. The fluxion of $\sqrt[3]{x}$ or $x^{\frac{1}{3}}$ is

18. The fluxion of $\sqrt[3]{x^2}$ or $x^{\frac{2}{3}}$ is

19. The fluxion of $\sqrt{x^3}$ or $x^{\frac{3}{2}}$ is

20. The fluxion of $\sqrt[4]{x^3}$ or $x^{\frac{3}{4}}$ is

21. The fluxion of $\sqrt[3]{x^4}$ or $x^{\frac{4}{3}}$ is

22. The fluxion of $\sqrt{(a^2 + x^2)}$ or $(a^2 + x^2)^{\frac{1}{2}}$ is

23. The fluxion of $\sqrt{(a^2 - x^2)}$ or $(a^2 - x^2)^{\frac{1}{2}}$ is

24. The fluxion of $\sqrt{(2rx - xx)}$ or $(2rx - xx)^{\frac{1}{2}}$ is

25. The fluxion of $\dfrac{1}{\sqrt{(a^2 - x^2)}}$ or $(a^2 - x^2)^{-\frac{1}{2}}$ is

26. The fluxion of $(ax - xx)^{\frac{1}{3}}$ is

27. The fluxion of $2x\sqrt{a^2 \pm x^2}$ is

28. The fluxion of $(a^2 - x^2)^{\frac{3}{2}}$ is

29. The fluxion of $\sqrt{xz}$ or $(xz)^{\frac{1}{2}}$ is

30. The fluxion of $\sqrt{xz - zz}$ or $(xz - zz)^{\frac{1}{2}}$ is

31. The fluxion of $-\dfrac{1}{a\sqrt{x}}$ or $-\frac{1}{a}x^{\frac{1}{2}}$ is

32. The fluxion of $\dfrac{ax^3}{a + x}$ is

33. The fluxion of $\dfrac{x^m}{y^n}$ is

34. The fluxion of $\dfrac{x + y + z}{x + y}$ is

35. The fluxion of $\dfrac{c}{xx}$ is

36. The fluxion of $\frac{3x}{a-x}$ is

37. The fluxion of $\frac{z}{x+z}$ is

38. The fluxion of $\sqrt[3]{(a+bx+cx^2+dx^3)}$ is

39. The fluxion of $\sqrt[n]{(a+bx+cx^2+\&c. \text{ to } mx^n)}$ is

40. The fluxion of $\frac{axy^2}{\quad}$ is

41. The fluxion of $\frac{\text{[illegible]}}{\sqrt{(x^2-y^2)}}$ is

42. The fluxion of the hyp. log. of ax is

43. The fluxion of the hyp. log. of $1+x$ is

44. The fluxion of the hyp. log. of $1-x$ is

45. The fluxion of the hyp. log. of x^2 is

46. The fluxion of the hyp. log. of $\sqrt{z}$ is

47. The fluxion of the hyp. log. of x^m is

48. The fluxion of the hyp. log. of $\frac{2}{x^2}$ is

49. The fluxion of the hyp. log. of $\frac{1+x}{1-x}$ is

50. The fluxion of the hyp. log. of $\frac{1-x}{1+x}$ is

51. The fluxion of c^x is

52. The fluxion of 10^x is

53. The fluxion of $(a+c)^x$ is

54. The fluxion of 100^{xy} is

55. The fluxion of x^z is

56. The fluxion of y^{10x} is

57. The fluxion of x^x is

58. The fluxion of $(xy)^{xz}$ is

59. The fluxion of $x\dot{y}$ is

60. The fluxion of $\dot{x}\dot{y}^2$ is

61. The second fluxion of xy is

62. The second fluxion of xy, when $\dot{x}$ is constant, is

63. The second fluxion of x^n is

64. The third fluxion of x^n, when $\dot{x}$ is constant, is

65. The third fluxion of xy is

THE INVERSE METHOD, OR THE FINDING OF FLUENTS.

37. It has been observed, that a Fluent, or Flowing Quantity, is the variable quantity which is considered as increasing or decreasing. Or, the fluent of a given fluxion, is such a quantity, that its fluxion, found according to the foregoing rules, shall be the same as the fluxion given or proposed.

38. It may further be observed, that Contemporary Fluents, or Contemporary Fluxions, are such as flow together, or for the same time.—When contemporary fluents are always equal, or in any constant ratio; then also are their fluxions respectively either equal, or in that same constant ratio. That is, if $x = y$, then is $\dot{x} = \dot{y}$; or if $x : y :: n : 1$, then is $\dot{x} : \dot{y} :: n : 1$; or if $x = ny$, then is $\dot{x} = n\dot{y}$.

39. It is easy to find the fluxions to all the given forms of fluents; but, on the contrary, it is difficult to find the fluents of many given fluxions; and indeed there are numberless cases in which this cannot at all be done, excepting by the quadrature and rectification of curve lines, or by logarithms, or by infinite series. For it is only in certain particular forms and cases that the fluents of given fluxions can be found; there being no method of performing this universally, *a priori*, by a direct investigation, like finding the fluxion of a given fluent quantity. We can only therefore lay down a few rules for such forms of fluxions as we know, from the direct method, belong to such and such kinds of flowing quantities: and these rules, it is evident, must chiefly consist in performing such operations as are the reverse of those by which the fluxions are found of given fluent quantities. The principal cases of which are as follow.

40. *To find the Fluent of a Simple Fluxion; or of that in which there is no variable quantity, and only one fluxional quantity.*

This is done by barely substituting the variable or flowing quantity instead of its fluxion; being the result or reverse of the notation only.—Thus,

The fluent of $a\dot{x}$ is ax.

The fluent of $a\dot{y} + 2\dot{y}$ is $ay + 2y$.

The fluent of $\dot{\phi}\sqrt{a^2 + x^2}$ is $\sqrt{a^2 + x^2}$.

41. *When any Power of a flowing quantity is Multiplied by the Fluxion of the Root:*

Then, having substituted, as before, the flowing quantity, for its fluxion, divide the result by the new index of the power. Or, which is the same thing, take out, or divide by, the fluxion of the root; add 1 to the index of the power; and divide by the index so increased. Which is the reverse of the 1st rule for finding fluxions.

So if the fluxion proposed be - - $3x^5\dot{x}$.
Leave out, or divide by, $\dot{x}$, then it is - $3x^5$;
add 1 to the index, and it is - - $3x^6$;
divide by the index 6, and it is - - $\frac{3}{6}x^6$ or $\frac{1}{2}x^6$,
which is the fluent of the proposed fluxion $3x^5\dot{x}$.

In like manner,

The fluent of $2ax\dot{x}$ is ax^2.

The fluent of $3x^2\dot{x}$ is x^3.

The fluent of $4x^{\frac{1}{2}}\dot{x}$ is $\frac{8}{3}x^{\frac{3}{2}}$.

The fluent of $2y^{\frac{3}{4}}\dot{y}$ is $\frac{8}{7}y^{\frac{7}{4}}$.

The fluent of $az^{\frac{5}{6}}\dot{z}$ is $\frac{6}{11}az^{\frac{11}{6}}$.

The fluent of $x^{\frac{1}{2}}\dot{x} + 3y^{\frac{2}{3}}\dot{y}$ is $\frac{2}{3}x^{\frac{3}{2}} + \frac{9}{5}y^{\frac{5}{3}}$.

The fluent of $x^{n-1}\dot{x}$ is $\frac{1}{n}x^n$.

The fluent of $ny^{n-1}\dot{y}$ is

The fluent of $\frac{\dot{z}}{z^2}$, or $z^{-2}\dot{z}$ is

The fluent of $\frac{a\dot{y}}{y^n}$ is

The fluent of $(a+x)^4\dot{x}$ is

The fluent of $(a^4+y^4)y^3\dot{y}$ is

The fluent of $(a^3+z^3)^4z^2\dot{z}$ is

The fluent of $(a^n+x^n)^m x^{n-1}\dot{x}$ is

The fluent of $(a^2+y^2)^3y\dot{y}$ is

The fluent of $\dfrac{z\dot{z}}{\sqrt{(a^2+z^2)}}$ is

The fluent of $\dfrac{\dot{x}}{\sqrt{(a-x)}}$ is

42. *When the Root under a Vinculum is a Compound Quantity; and the Index of the part or factor Without the Vinculum, increased by* 1, *is some Multiple of that Under the Vinculum:*

Put a single variable letter for the compound root; and substitute its power and fluxion instead of those of the same value, in the given quantity; so will it be reduced to a simpler form, to which the preceding rule can then be applied.

Thus, if the given fluxion be $\dot{y} = (a^2+x^2)^{\frac{2}{3}}x^3\dot{x}$, where 3, the index of the quantity without the vinculum, increased by 1, making 4, which is just the double of 2, the exponent of x^2 within the vinculum: therefore, putting $\dot{z} = a^2+x^2$, thence $x^2 = z - a^2$, the fluxion of which is $2x\dot{x} = \dot{z}$; hence then $x^3\dot{x} = \frac{1}{2}x^2\dot{z} = \frac{1}{2}\dot{z}(z-a^2)$, and the given fluxion $\dot{y}$, or $(a^2+x^2)^{\frac{2}{3}}x^3\dot{x}$, is $= \frac{1}{2}z^{\frac{2}{3}}\dot{z}(z-a^2)$ or $= \frac{1}{2}z^{\frac{5}{3}}\dot{z} - \frac{1}{2}a^2z^{\frac{2}{3}}\dot{z}$; and hence the fluent y is $= \frac{3}{16}z^{\frac{8}{3}} - \frac{3}{10}a^2z^{\frac{5}{3}} = 3z^{\frac{5}{3}}(\frac{1}{16}z - \frac{1}{10}a^2)$. Or, by substituting the value of z instead of it, the same fluent is $3(a^2+x^2)^{\frac{5}{3}} \times (\frac{1}{16}x^2 - \frac{3}{80}a^2)$, or $\frac{3}{16}(a^2+x^2) \times (x^2 - \frac{3}{5}a^2)$.

In like manner for the following examples.

To find the fluent of $\overline{\sqrt{a+cx}} \times x^3\dot{x}$.

To find the fluent of $(a+cx)^{\frac{3}{4}}x^2\dot{x}$.

To find the fluent of $(a+cx^2)^{\frac{1}{3}} \times dx^3\dot{x}$.

To find the fluent of $\dfrac{cz\dot{z}}{\sqrt{a+z}}$ or $(a+z)^{\frac{1}{2}}cz\dot{z}$.

To find the fluent of $\frac{cz^{3n-1}\dot{z}}{\sqrt{a+z^n}}$ or $(a+z^n)^{-\frac{1}{2}}cz^{3n-1}\dot{z}$.

To find the fluent of $\frac{\dot{z}\sqrt{a^2-z^2}}{z^6}$ or $(a^2+z^2)^{\frac{1}{2}}z^{-6}\dot{z}$.

To find the fluent of $\frac{\dot{x}\sqrt{a-x^n}}{x^{\frac{7}{2}n+1}}$ or $(a-x^n)^{\frac{1}{2}}x^{-\frac{7}{2}n-1}\dot{x}$.

43. *When there are several Terms, involving Two or more Variable Quantities, having the Fluxion of each Multiplied by the other Quantity or Quantities.*

Take the fluent of each term, as if there were only one variable quantity in it, namely, that whose fluxion is contained in it, supposing all the others to be constant in that term; then, if the fluents of all the terms, so found, be the very same quantity in all of them, that quantity will be the fluent of the whole. Which is the reverse of the 5th rule for finding fluxions: Thus, if the given fluxion be $\dot{x}y+x\dot{y}$, then the fluent of $\dot{x}y$ is xy, supposing y constant: and the fluent of $x\dot{y}$ is also xy, supposing x constant: therefore xy is the required fluent of the given fluxion $\dot{x}y+x\dot{y}$.

In like manner,

The fluent of $\dot{x}yz+x\dot{y}z+xy\dot{z}$ is xyz.

The fluent of $2xy\dot{x}+x^2\dot{y}$ is x^2y.

The fluent of $\frac{1}{2}x^{-\frac{1}{2}}\dot{x}y^2+2x^{\frac{1}{2}}y\dot{y}$ is

The fluent of $\frac{\dot{x}y-x\dot{y}}{y^2}$ or $\frac{\dot{x}}{y}-\frac{x\dot{y}}{y^2}$ is

The fluent of $\frac{2ax\dot{x}y^{\frac{1}{2}}-\frac{1}{2}ax^2y^{-\frac{1}{2}}\dot{y}}{y}$ or $\frac{2ax\dot{x}}{\sqrt{y}}-\frac{ax^2\dot{y}}{2y\sqrt{y}}$ is

44. *When the given Fluxional Expression is in this Form* $\frac{\dot{x}y-x\dot{y}}{y^2}$, *namely, a fraction, including Two Quantities, being the Fluxion of the former of them drawn into the latter, minus the Fluxion of the latter drawn into the former, and divided by the Square of the latter.*

Then, the fluent is the fraction $\frac{x}{y}$, or the former quantity

divided by the latter, by the reverse of rule 4, of finding fluxions. That is,

The fluent of $\frac{\dot{x}y - x\dot{y}}{y^2}$ is $\frac{x}{y}$. And, in like manner,

The fluent of $\frac{2x\dot{x}y^2 - 2x^2y\dot{y}}{y^4}$ is $\frac{x^2}{y^2}$.

Though, indeed, the examples of this case may be performed by the foregoing one. Thus, the given fluxion $\frac{\dot{x}y - x\dot{y}}{y^2}$ reduces to $\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2}$, or $\frac{\dot{x}}{y} - x\dot{y}y^{-2}$; of which,

the fluent of $\frac{\dot{x}}{y}$ is $\frac{x}{y}$ supposing y constant; and

the fluent of $-x\dot{y}y^{-2}$ is also xy^{-1} or $\frac{x}{y}$, when x is constant;

therefore, by that case, $\frac{x}{y}$ is the fluent of the whole $\frac{\dot{x}y - x\dot{y}}{y^2}$.

45. *When the Fluxion of a Quantity is Divided by the Quantity itself:*

Then the fluent is equal to the hyperbolic logarithm of that quantity; or, which is the same thing, the fluent is equal to 2·30258509 multiplied by the common logarithm of the same quantity, by rule 6, for finding fluxions.

So, the fluent of $\frac{\dot{x}}{x}$ or $x^{-1}\dot{x}$, is the hyp. log. of x.

The fluent of $\frac{2\dot{x}}{x}$ is $2 \times$ hyp. log. of x, or $=$ hyp. log. x^2.

The fluent of $\frac{a\dot{x}}{x}$, is $a \times$ hyp. log. x, or $=$ hyp. log. of x^a.

The fluent of $\frac{\dot{x}}{a+x}$, is

The fluent of $\frac{3x^2\dot{x}}{a+x^3}$, is

46. *Many fluents may be found by the Direct Method thus:*

Take the fluxion again of the given fluxion, or the second

fluxion of the fluent sought; into which substitute $\frac{\dot{x}^2}{x}$ for $\ddot{x}$, $\frac{\dot{y}^2}{y}$ for $\ddot{y}$, &c.; that is, make x, $\dot{x}$, $\ddot{x}$, as also y, $\dot{y}$, $\ddot{y}$, &c. to be in continual proportion, or so that $x : \dot{x} :: \dot{x} : \ddot{x}$, and $y : \dot{y} :: \dot{y} : \ddot{y}$, &c.; then divide the square of the given fluxional expression by the second fluxion, just found, and the quotient will be the fluent required in many cases.

Or the same rule may be otherwise delivered thus:

In the given fluxion $\dot{\text{F}}$, write x for $\dot{x}$, y for $\dot{y}$, &c. and call the result G, taking also the fluxion of this quantity, $\dot{\text{G}}$; then make $\dot{\text{G}} : \dot{\text{F}} :: \text{G} : \text{F}$; so shall the fourth proportional F be the fluent sought, in many cases.

It may be proved if this be the true fluent, by taking the fluxion of it again, which if it agree with the proposed fluxion, will show that the fluent is right; otherwise, it is wrong.

EXAMPLES.

Exam. 1. Let it be required to find the fluent of $nx^{n-1}\dot{x}$.

Here $\dot{\text{F}} = nx^{n-1}\dot{x}$. Write x for $\dot{x}$, then $nx^{n-1}x$ or $nx^n = \text{G}$; the fluxion of this is $\dot{\text{G}} = n^2x^{n-1}\dot{x}$; therefore $\dot{\text{G}} : \dot{\text{F}} :: \text{G} : \text{F}$, becomes $n^2x^{n-1}\dot{x} : nx^{n-1}\dot{x} :: nx^n : x^n = \text{F}$, the fluent sought.

Exam. 2. To find the fluent of $\dot{x}y + x\dot{y}$.

Here $\dot{\text{F}} = \dot{x}y + x\dot{y}$; then writing x for $\dot{x}$, and y for $\dot{y}$, it is $xy + xy$ or $2xy = \text{G}$; hence $\dot{\text{G}} = 2\dot{x}y + 2x\dot{y}$; then $\dot{\text{G}} : \dot{\text{F}} :: \text{G} : \text{F}$, becomes $2\dot{x}y + 2x\dot{y} : \dot{x}y + x\dot{y} :: 2xy : xy = \text{F}$, the fluent sought.

47. *To find Fluents by means of a Table of Forms of Fluxions and Fluents.*

In the following Table are contained the most usual forms of fluxions that occur in the practical solution of problems, with their corresponding fluents set opposite to them; by means of which, namely, by comparing any proposed fluxion with the corresponding form in the table, the fluent of it will be found.

Forms.	*Fluxions.*	*Fluents.*
1	$x^{n-1}\dot{x}$	$\frac{1}{n}x^n$
2	$(a \pm x^n)^{m-1}x^{n-1}\dot{x}$	$\pm\frac{1}{mn}(a \pm x^n)^m$
3	$\frac{x^{mn-1}\dot{x}}{(a \pm x^n)^{m+1}}$	$\frac{1}{mna} \times \frac{x^{mn}}{(a \pm x^n)^m}$
4	$\frac{(a \pm x^n)^{m-1}\dot{x}}{x^{mn+1}}$	$\frac{-1}{mna} \times \frac{(a \pm x^n)^m}{x^{mn}}$
5	$(my\dot{x} + nx\dot{y}) \times x^{m-1}y^{n-1}$, or $(\frac{m\dot{x}}{x} + \frac{n\dot{y}}{y})x^m y^n$	$x^m y^n$
6	$mx^{m-1}\dot{x}y^n z^r + nx^m y^{n-1}\dot{y}z^r + rx^m y^n z^{r-1}\dot{z}$, or $(m\dot{x}yz + nx\dot{y}z + rxy\dot{z})\, x^{m-1}y^{n-1}z^{r-1}$, or $(\frac{m\dot{x}}{x} + \frac{n\dot{y}}{y} + \frac{r\dot{z}}{z})x^m y^n z^r$,	$x^m y^n z^r$
7	$\frac{\dot{x}}{x}$ or $x^{-1}\dot{x}$	log. of x
8	$\frac{x^{n-1}\dot{x}}{a \pm x^n}$	$\pm\frac{1}{n}$ log. of $a \pm x^n$
9	$\frac{x^{-1}\dot{x}}{a \pm x^n}$	$\frac{1}{na}$ log. of $\frac{x^n}{a \pm x^n}$
10	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{a - x^n}$	$\frac{1}{n\sqrt{a}}$ log. of $\frac{\sqrt{a} + \sqrt{x^n}}{\sqrt{a} - \sqrt{x^n}}$
11	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{a + x^n}$	$\frac{2}{n\sqrt{a}} \times$ arc. to tan. $\sqrt{\frac{x^n}{a}}$, or $\frac{1}{n\sqrt{a}} \times$ arc to cosine $\frac{a - x^n}{a + x^n}$
12	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{(\pm a + x^n)}}$	$\frac{2}{n}$log.of $\sqrt{x^n} + \sqrt{(\pm a + x^n)}$

Forms.	*Fluxions.*	*Fluents.*
13	$\dfrac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{(a-x^n)}}$	$\dfrac{2}{n}\times$ arc to sin. $\sqrt{\dfrac{x^n}{a}}$, or $\dfrac{1}{n}\times$ arc to vers. $\dfrac{2x^n}{a}$
14	$\dfrac{x^{-1}\dot{x}}{\sqrt{(a\pm x^n)}}$	$\dfrac{1}{n\sqrt{a}}$ log. of $\dfrac{\sqrt{a}-\sqrt{(a\pm x^n)}}{\sqrt{a}+\sqrt{(a\pm x^n)}}$
15	$\dfrac{x^{-1}\dot{x}}{\sqrt{-a+x^n}}$	$\dfrac{2}{n\sqrt{a}}\times$ arc to secant $\sqrt{\dfrac{x^n}{a}}$, or $\dfrac{1}{n\sqrt{a}}\times$ arc to cosin. $\dfrac{2a-x^n}{x^n}$
16	$\dot{x}\sqrt{dx-x^2}$	$\frac{1}{2}$ circ. seg. to diam. d and vers. x
17	$2\dot{x}\sqrt{(a^2-x^2)}$	circ. zone, rad. a, and height from centre x.
18	$c^{nx}\dot{x}$	$\dfrac{c^{nx}}{n\log. c}$
19	$\dot{x}y^x\log. y+xy^{x-1}\dot{y}$	y^x
20	$x^{\frac{1}{2}}\dot{x}\sqrt{(bx\pm a)}$	$+\dfrac{x^{\frac{1}{2}}(2bx\pm a)\sqrt{(bx\pm a)}}{4b}-\dfrac{a^2}{4b\sqrt{b}}+\log.\left\{\sqrt{bx}+\sqrt{(bx\pm a)}\right\}$
21	$x^{\frac{1}{2}}\dot{x}\sqrt{(a-bx)}$	$+\dfrac{x^{\frac{1}{2}}(2bx-a)\sqrt{(a-bx)}}{4b}+\dfrac{a^2}{4b\sqrt{b}}\times$ arc. tang. $\sqrt{\dfrac{bx}{a-bx}}$
22	$\dfrac{\dot{x}\sqrt{(bx\pm a)}}{x^{\frac{1}{2}}}$	$+x^{\frac{1}{2}}\sqrt{(bx\pm a)}\pm\dfrac{a}{\sqrt{b}}\times\log.\left\{\sqrt{bx}+\sqrt{(bx\pm a)}\right\}$
23	$\dfrac{\dot{x}\sqrt{(a-bx)}}{x^{\frac{1}{2}}}$	$+x^{\frac{1}{2}}\sqrt{(a-bx)}+\dfrac{a}{\sqrt{b}}\times$ arc. tang. $\sqrt{-\dfrac{bx}{a-bx}}$
24	$\dfrac{\dot{x}\sqrt{(a\pm bx)}}{x}$	$2\sqrt{(a\pm bx)}-2a^{\frac{1}{2}}\times\log.\dfrac{\sqrt{a}+\sqrt{(a\pm bx)}}{\sqrt{x}}$
25	$\dfrac{\dot{x}\sqrt{(bx-a)}}{x}$	$2\sqrt{(bx-a)}-2a^{\frac{1}{2}}\times$ arc. tang. $\sqrt{\dfrac{bx-a}{a}}$

Forms.	*Fluxions.*	*Fluents.*
26	$\frac{\dot{x}\sqrt{(a \pm bx^2)}}{x}$	$= + \sqrt{(a \pm bx^2)} - a^{\frac{1}{2}} \times \log. \frac{\sqrt{a} + \sqrt{(a \pm bx^2)}}{x}$
27	$\frac{\dot{x}\sqrt{(bx^2 - a)}}{x}$	$= + \sqrt{(bx^2 - a)} - a^{\frac{1}{2}} \times \text{arc. tang.} \sqrt{\frac{bx^2 - a}{a}}$
28	$\frac{\dot{x}}{a + bx + cx^2}$	$= \frac{2}{\sqrt{(4ac - b^2)}} \times \text{arc. tan.} \frac{b + 2cx}{\sqrt{(4ac - b^2)}}.$
29	$\frac{\dot{x}}{a + bx - cx^2}$	$= \frac{2}{\sqrt{(4ac + b^2)}} \times \log. \frac{\sqrt{(4ac + b^2)} - (b - 2cx)}{\sqrt{(a + bx - cx^2)}}.$
30	$\frac{\dot{x}}{x(a + bx + cx^2)}$	$= \left\{ \begin{array}{l} - \frac{1}{a} \times \log. \frac{\sqrt{(a + bx + cx^2)}}{x}. \\ - \frac{b}{a\sqrt{(4ac - b^2)}} \times \text{arc. tan.} \frac{b + 2cx}{\sqrt{(4ac - b^2)}}. \end{array} \right.$
31	$\frac{\dot{x}}{x(a + bx - cx^2)}$	$= \left\{ \begin{array}{l} - \frac{1}{a} \times \log. \frac{\sqrt{(a + bx - cx^2)}}{x}. \\ - \frac{b}{a\sqrt{(4ac + b^2)}} \times \log. \frac{\sqrt{(4ac + b^2)} - (b - 2cx)}{\sqrt{(a + bx - cx^2)}}. \end{array} \right.$
32	$\frac{x\dot{x}}{a + bx + cx^2}$	$= \left\{ \begin{array}{l} + \frac{1}{2c} \times \log. (a + bx + cx^2) \\ - \frac{b}{c\sqrt{(4ac - b^2)}} \times \text{arc. tan.} \frac{b + 2cx}{\sqrt{(4ac - b^2)}}. \end{array} \right.$
33	$\frac{x\dot{x}}{a + bx - cx^2}$	$= \left\{ \begin{array}{l} - \frac{1}{2c} \times \log. (a + bx - cx^2) \\ + \frac{b}{c\sqrt{(4ac + b^2)}} \times \log. \frac{\sqrt{(4ac + b^2)} - (b - 2cx)}{\sqrt{(a + bx - cx^2)}}. \end{array} \right.$
34	$\dot{x}\sqrt{(a + bx + cx^2)}$	$= \left\{ \begin{array}{l} + \frac{(2cx + b)\sqrt{(a + bx + cx^2)}}{4c} + \frac{4ac - b^2}{8c\sqrt{c}} \times \\ \log. \left\{ 2cx + b + 2c^{\frac{1}{2}}\sqrt{(a + bx + cx^2)} \right\}. \end{array} \right.$
35	$\dot{x}\sqrt{(a + bx - cx^2)}$	$= \left\{ \begin{array}{l} + \frac{(2cx - b)\sqrt{(a + bx - cx^2)}}{4} + \frac{4ac + b^2}{8c\sqrt{c}} \times \\ \text{arc. tan.} \frac{2cx - b}{2c^{\frac{1}{2}}\sqrt{(a + bx - cx^2)}}. \end{array} \right.$
36	$\frac{(A + Bx)\dot{x}}{a + bx + cx^2}$	$= \left\{ \begin{array}{l} + \frac{B}{2c} \times \log. (a + bx + cx^2) \\ + \frac{2cA - bB}{c\sqrt{(4ac - b^2)}} \times \text{arc. tan.} \frac{b + 2cx}{\sqrt{(4ac - b^2)}}. \end{array} \right.$

Forms.	*Fluxions.*	*Fluents.*
37	$\dfrac{(A + Bx)\dot{x}}{a + bx - cx^2}$	$= \begin{cases} -\dfrac{B}{2c} \times \log. (a + bx - cx^2) \\ +\dfrac{2cA + bB}{c\sqrt{(4ac + b^2)}} \times \log. \dfrac{\sqrt{(4ac + b^2)} - (b - 2cx)}{\sqrt{(a + bx - cx^2)}}. \end{cases}$
38	$\dfrac{\dot{x}}{\sqrt{(a + bx + cx^2)}}$	$= \begin{cases} +\dfrac{1}{\sqrt{c}} \times \log. \{2cx + b + 2c^{\frac{1}{2}} \\ \quad \sqrt{(a + bx + cx^2)}\}. \end{cases}$
39	$\dfrac{\dot{x}}{\sqrt{(a + bx - cx^2)}}$	$= \dfrac{1}{\sqrt{c}} \times \text{arc. tan.} \dfrac{2cx - b}{2c^{\frac{1}{2}}\sqrt{(a + bx - cx^2)}}.$
40	$\dfrac{\dot{x}}{x\sqrt{(a + bx + cx^2)}}$	$= -\dfrac{1}{\sqrt{a}} \times \log. \left\{\dfrac{2a + bx + 2a^{\frac{1}{2}}\sqrt{(a + bx + cx^2)}}{x}.\right\}$
41	$\dfrac{\dot{x}}{x\sqrt{(-a + bx + cx^2)}}$	$= +\dfrac{1}{\sqrt{a}} \times \text{arc. tan.} \dfrac{2a^{\frac{1}{2}}\sqrt{(-a + bx + cx^2)}}{2a - bx}.$

Note. The logarithms, in the above forms, are the hyperbolic ones, which are found by multiplying the common logarithms by 2·302585092994. And the arcs, whose sine, or tangent, &c. are mentioned, have the radius 1, and are those in the common tables of sines, tangents, and secants. Also, the numbers m, n, &c, are to be some real quantities, as the forms fail when $m = 0$, or $n = 0$, &c.

The Use of the Foregoing Table of Forms of Fluxions and Fluents.

48. In using the foregoing table, it is to be observed, that the first column serves only to show the number of the form; in the second column are the several forms of fluxions, which are of different kinds or classes; and in the third or last column, are the corresponding fluents.

The method of using the table, is this. Having any fluxion given, to find its fluent: First, Compare the given fluxion with the several forms of fluxions in the second column of the table, till one of the forms be found that agrees with it; which is done by comparing the terms of the given fluxion with the like parts of the tabular fluxion, namely, the radical quantity of the one, with that of the other; and the exponents of the variable quantities of each, both within and without the vinculum; all which, being found to agree

or correspond, will give the particular values of the general quantities in the tabular form: then substitute these particular values in the general or tabular form of the fluent, and the result will be the particular fluent of the given fluxion; after it is multiplied by any co-efficient the proposed fluxion may have.

EXAMPLES.

EXAM. 1. To find the fluent of the fluxion $3x^{\frac{5}{3}}\dot{x}$.

This is found to agree with the first form. And, by comparing the fluxions, it appears that $x = x$, and $n - 1 = \frac{5}{3}$, or $n = \frac{8}{3}$; which being substituted in the tabular fluent, or $\frac{1}{n}x^n$, gives, after multiplying by 3 the co-efficient, $3 \times \frac{3}{8}x^{\frac{8}{3}}$, or $\frac{9}{8}x^{\frac{8}{3}}$, for the fluent sought.

EXAM. 2. To find the fluent of $5x^2\dot{x}\sqrt{c^3 - x^3}$, or $5x^2\dot{x}(c^3 - x^3)^{\frac{1}{2}}$.

This fluxion, it appears, belongs to the 2d tabular form: for $a = c^3$, and $-x^n = -x^3$, and $n = 3$ under the vinculum, also $m - 1 = \frac{1}{2}$, or $m = \frac{3}{2}$, and the exponent ${}^{n-1}$ of x^{n-1} without the vinculum, by using 3 for n, is $n - 1 = 2$, which agrees with x^2 in the given fluxion: so that all the parts of the form are found to correspond. Then substituting these values into the general fluent, $-\frac{1}{mn}(a - x^n)^m$, it becomes $-\frac{5}{3} \times \frac{2}{3}(c^3 - x^3)^{\frac{3}{2}} = -\frac{10}{9}(c^3 - x^3)^{\frac{3}{2}}$.

EXAM. 3. To find the fluent of $\frac{x^2\dot{x}}{1 + x^3}$.

This is found to agree with the 8th form; where - - - $\pm x^n = + x^3$ in the denominator, or $n = 3$; and the numerator x^{n-1} then becomes x^2, which agrees with the numerator in the given fluxion; also $a = 1$. Hence then, by substituting in the general or tabular fluent, $\frac{1}{n}$ log. of $a + x^n$, it becomes $\frac{1}{3}$ log. $1 + x^3$.

EXAM. 4. To find the fluent of $ax^4\dot{x}$.

EXAM. 5. To find the fluent of $2(10 + x^3)^{\frac{2}{3}}x\dot{x}$.

EXAM. 6. To find the fluent of $\frac{a\dot{x}}{(c^2 + x^2)^{\frac{3}{2}}}$.

EXAM. 7. To find the fluent of $\frac{3x^2\dot{x}}{(a - x)^4}$

EXAM. 8. To find the fluent of $\frac{c^2-x^2}{x^5}\dot{x}$.

EXAM. 9. To find the fluent of $\frac{1+3x}{2x^4}\dot{x}$.

EXAM. 10. To find the fluent of $(\frac{3\dot{x}}{x}+\frac{2\dot{y}}{y})x^3y^2$.

EXAM. 11. To find the fluent of $(\frac{\dot{x}}{x}+\frac{\dot{y}}{3y})xy^{\frac{1}{3}}$.

EXAM. 12. To find the fluent of $\frac{3\dot{x}}{ax}$ or $\frac{3}{a}x^{-1}\dot{x}$.

EXAM. 13. To find the fluent of $\frac{a\dot{x}}{3-2x}$.

EXAM. 14. To find the fluent of $\frac{3\dot{x}}{2x-x^2}$ or $\frac{3x^{-1}\dot{x}}{2-x}$.

EXAM. 15. To find the fluent of $\frac{2\dot{x}}{x-3x^3}$ or $\frac{2x^{-1}\dot{x}}{1-3x^2}$.

EXAM. 16. To find the fluent of $\frac{3x\dot{x}}{1-x^4}$.

EXAM. 17. To find the fluent of $\frac{ax^{\frac{3}{2}}\dot{x}}{2-x^5}$.

EXAM. 18. To find the fluent of $\frac{2x\dot{x}}{1+x^4}$.

EXAM. 19. To find the fluent of $\frac{ax^{\frac{3}{2}}\dot{x}}{2+x^5}$.

EXAM. 20. To find the fluent of $\frac{3x\dot{x}}{\sqrt{1+x^4}}$.

EXAM. 21. To find the fluent of $\frac{a\dot{x}}{\sqrt{x^2-4}}$.

EXAM. 22. To find the fluent of $\frac{3x\dot{x}}{\sqrt{1-x^4}}$.

EXAM. 23. To find the fluent of $\frac{a\dot{x}}{\sqrt{4-x^2}}$.

EXAM. 24. To find the fluent of $\frac{2x^{-1}\dot{x}}{\sqrt{1-x^2}}$.

EXAM. 25. To find the fluent of $\frac{a\dot{x}}{\sqrt{ax^2+x^{\frac{1}{2}}}}$.

EXAM. 26. To find the fluent of $\frac{2x^{-1}\dot{x}}{\sqrt{x^2-1}}$.

EXAM. 27. To find the fluent of $\frac{a\dot{x}}{\sqrt{(x^{\frac{1}{2}} - ax)^2}}$

EXAM. 28. To find the fluent of $2\dot{x}\sqrt{2x - x^2}$.

EXAM. 29. To find the fluent of $a^x\dot{x}$.

EXAM. 30. To find the fluent of $3a^{2x}\dot{x}$.

EXAM. 31. To find the fluent of $3z^x\dot{x}$ log. $z + 3xz^{x-1}\dot{z}$.

EXAM. 32. To find the fluent of $(1 + x^3)\,x\dot{x}$.

EXAM. 33. To find the fluent of $(2 + x^4)x^{\frac{3}{2}}\dot{x}$.

EXAM. 34. To find the fluent of $x^2\dot{x}\sqrt{(a^2 + x^2)}$.

To find Fluents by Infinite Series.

49. When a given fluxion, whose fluent is required, is so complex, that it cannot be made to agree with any of the forms in the foregoing table of cases, nor made out from the general rules before given; recourse may then be had to the method of infinite series; which is thus performed:

Expand the radical or fraction, in the given fluxion, into an infinite series of simple terms, by the methods given for that purpose in books of algebra; viz. either by division or extraction of roots, or by the binomial theorem, &c.; and multiply every term by the fluxional letter, and by such simple variable factor as the given fluxional expression may contain. Then take the fluent of each term separately, by the foregoing rules, connecting them all together by their proper signs; and the series will be the fluent sought, after it is multiplied by any constant factor or co-efficient which may be contained in the given fluxional expression.

50. It is to be noted, however, that the quantities must be so arranged, as that the series produced may be a converging one, rather than diverging: and this is effected by placing the greater terms foremost in the given fluxion. When these are known or constant quantities, the infinite series will be an ascending one; that is, the powers of the variable quantity will ascend or increase; but if the variable quantity be set foremost, the infinite series produced will be a descending one, or the powers of that quantity will decrease always more and more in the succeeding terms, or increase in the denominators of them, which is the same thing.

For example, to find the fluent of $\frac{1 - x}{1 + x - x^2}\dot{x}$.

Here, by dividing the numerator by the denominator, the proposed fluxion becomes $\dot{x} - 2x\dot{x} + 3x^2\dot{x} - 5x^3\dot{x} + 8x^4\dot{x} -$ &c.; then the fluents of all the terms being taken, give $x - x^2 + x^3 - \frac{5}{4}x^4 + \frac{8}{5}x^5 -$ &c. for the fluent sought.

Again, to find the fluent of $\dot{x}\sqrt{1 - x^2}$.

Here, by extracting the root, or expanding the radical quantity $\sqrt{1 - x^2}$, the given fluxion becomes - - - - $\dot{x} - \frac{1}{2}x^2\dot{x} - \frac{1}{8}x^4\dot{x} - \frac{1}{16}x^6\dot{x} -$ &c. Then the fluents of all the terms, being taken, give $x - \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{1}{112}x^7 -$ &c. for the fluent sought.

OTHER EXAMPLES.

EXAM. 1. To find the fluent of $\frac{bx\dot{x}}{a - x}$ both in an ascending and descending series.

EXAM. 2. To find the fluent of $\frac{b\dot{x}}{a + x}$ in both series.

EXAM. 3. To find the fluent of $\frac{3\dot{x}}{(a + x)^2}$.

EXAM. 4. To find the fluent of $\frac{1 - x^2 + 2x^4}{1 + x - x^2}\dot{x}$.

EXAM. 5. Given $\dot{z} = \frac{b\dot{x}}{a^2 + x^2}$, to find z.

EXAM. 6. Given $\dot{z} = \frac{a^2 + x^2}{a + x}\dot{x}$ to find z.

EXAM. 7. Given $\dot{z} = 3\dot{x}\sqrt{a + x}$, to find z.

EXAM. 8. Given $\dot{z} = 2\dot{x}\sqrt{a^2 + x^2}$, to find z.

EXAM. 9. Given $\dot{z} = 4\dot{x}\sqrt{a^2 - x^2}$, to find z.

EXAM. 10. Given $\dot{z} = \frac{5a\dot{x}}{[illegible]}$, to find z.

EXAM. 11. Given $\dot{z} = 2\dot{x}\sqrt[3]{a^3 - x^3}$, to find z.

EXAM. 12. Given $\dot{z} = \frac{3a\dot{x}}{\sqrt{ax - xx}}$, to find z.

EXAM. 13. Given $\dot{z} = 2\dot{x}\sqrt{x^3 + x^4 + x^5}$, to find z.

EXAM. 14. Given $\dot{z} = 5\dot{x}\sqrt{ax - xx}$, to find z.

For more on the finding of fluents, the student may consult the chapter on that subject in the third volume.

To Correct the Fluent of any Given Fluxion.

51. The fluxion found from a given fluent is always perfect and complete; but the fluent found from a given fluxion is not always so; as it often wants a correction, to make it contemporaneous with that required by the problem under consideration, &c.: for, the fluent of any given fluxion, as $\dot{x}$ may be either x, which is found by the rule, or it may be $x + c$, or $x - c$, that is x plus or minus some constant quantity c; because both x and $x \pm c$ have the same fluxion $\dot{x}$, and the finding of the constant quantity c, to be added or subtracted with the fluent as found by the foregoing rules, is called *correcting* the fluent.

Now this correction is to be determined from the nature of the problem in hand, by which we come to know the relation which the fluent quantities have to each other at some certain point or time. Reduce, therefore, the general fluential equation, supposed to be found by the foregoing rules, to that point or time; then if the equation be true, it is correct; but if not, it wants a correction; and the quantity of the correction, is the difference between the two general sides of the equation when reduced to that particular point. Hence the general rule for the correction is this:

Connect the constant, but indeterminate, quantity c, with one side of the fluential equation, as determined by the foregoing rules; then, in this equation, substitute for the variable quantities, such values as they are known to have at any particular state, place, or time; and then, from that particular state of the equation, find the value of c, the constant quantity of the correction.

EXAMPLES.

52. EXAM. 1. To find the correct fluent of $\dot{z} = ax^3\dot{x}$.

The general fluent is $z = ax^4$, or $z = ax^4 + c$, taking in the correction c.

Now, if it be known that z and x begin together, or that z is $= 0$, when $x = 0$; then writing 0 for both x and z, the general equation becomes $0 = 0 + c$, or $= c$; so that, the value of c being 0, the correct fluents are $z = ax^4$.

But if z be $= 0$, when x is $= b$, any known quantity; then substituting 0 for z, and b for x, in the general equation, it becomes $0 = ab^4 + c$, and hence we find $c = -ab^4$; which being written for c in the general fluential equation, it becomes $z = ax^4 - ab^4$, for the correct fluents.

Or, if it be known that z is $=$ some quantity d, when x is $=$ some other quantity as b; then substituting d for z, and b for x, in the general fluential equation $z = ax^4 + c$, it becomes $d = ab^4 + c$; and hence is deduced the value of the correction, namely, $c = d - ab^4$; consequently, writing this value for c in the general equation, it becomes - - - $z = ax^4 - ab^4 + d$, for the correct equation of the fluents in this case.

53. And hence arises another easy and general way of correcting the fluents, which is this: In the general equation of the fluents, write the particular values of the quantities which they are known to have at any certain time or position; then subtract the sides of the resulting particular equation from the corresponding sides of the general one, and the remainders will give the correct equation of the fluents sought.

So, the general equation being $z = ax^4$;
write d for z, and b for x, then $d = ab^4$;
hence, by subtraction, - $z - d = ax^4 - ab^4$,
or $z = ax^4 - ab^4 + d$, the correct fluents as before.

EXAM. 2. To find the correct fluents of $\dot{z} = 5x\dot{x}$; z being $= 0$ when x is $= a$.

EXAM. 3. To find the correct fluents of $\dot{z} = 3\dot{x}\sqrt{a + x}$; z and x being $= 0$ at the same time.

EXAM. 4. To find the correct fluent of $\dot{z} = \frac{2a\dot{x}}{a + x}$; supposing z and x to begin to flow together, or to be each $= 0$ at the same time.

EXAM. 5. To find the correct fluents of $\dot{z} = \frac{2\dot{x}}{a^2 + x^2}$; supposing z and x to begin together.

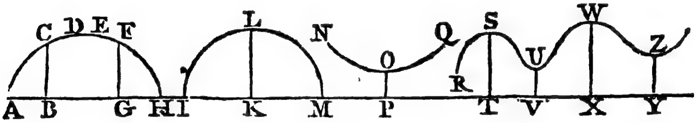
OF MAXIMA AND MINIMA; OR, THE GREATEST AND LEAST MAGNITUDE OF VARIABLE OR FLOWING QUANTITIES.

54. MAXIMUM, denotes the greatest state or quantity attainable in any given case, or the greatest value of a variable quantity: by which it stands opposed to Minimum, which is the least possible quantity in any case.

Thus, the expression or sum $a^2 + bx$, evidently increases as x, or the term bx, increases; therefore the given expression will be the greatest, or a maximum, when x is the greatest, or infinite: and the same expression will be a minimum, or the least, when x is the least, or nothing.

Again, in the algebraic expression $a^2 - bx$, where a and b denote constant or invariable quantities, and x a flowing or variable one, it is evident that the value of this remainder or difference, $a^2 - bx$, will increase, as the term bx, or as x, decreases; therefore the former will be the greatest, when the latter is the smallest; that is, $a^2 - bx$ is a maximum, when x is the least, or nothing at all; and the difference is the least, when x is the greatest.

55. Some variable quantities increase continually; and so have no maximum, but what is infinite. Others again decrease continually; and so have no minimum, but what is of no magnitude, or nothing. But, on the other hand, some variable quantities increase only to a certain finite magnitude, called their Maximum, or greatest state, and after that they decrease again. While others decrease to a certain finite magnitude, called their Minimum, or least state, and afterwards increase again. And lastly, some quantities have several maxima and minima.



Thus, for example, the ordinate BC of the parabola, or such-like curve, flowing along the axis AB from the vertex A, continually increases, and has no limit or maximum. And the ordinate GF of the curve EFH, flowing from E towards H, continually decreases to nothing when it arrives at the point H. But in the circle ILM, the ordinate only increases to a certain magnitude, namely, the radius, when it arrives at the middle as at KL, which is its maximum; and after that it decreases again to nothing, at the point M. And in the curve NOQ, the ordinate decreases only to the position OP, where it is least, or a minimum; and after that it continually increases towards Q. But in the curve RSU, &c, the ordinates have several maxima, as ST, WX, and several minima, as VU, YZ, &c.

56. Now, because the fluxion of a variable quantity, is the rate of its increase or decrease; and because the maximum

or minimum of a quantity neither increases nor decreases, at those points or states; therefore such maximum or minimum has no fluxion, or the fluxion is then equal to nothing. From which we have the following rule.

To find the Maximum or Minimum.

57. From the nature of the question or problem, find an algebraical expression for the value, or general state, of the quantity whose maximum or minimum is required; then take the fluxion of that expression, and put it equal to nothing; from which equation, by dividing by, or leaving out, the fluxional letter and other common quantities, and performing other proper reductions, as in common algebra, the value of the unknown quantity will be obtained, determining the point of the maximum or minimum.

So, if it be required to find the maximum state of the compound expression $100x - 5x^2 \pm c$, or the value of x when $100x - 5x^2 \pm c$ is a maximum. The fluxion of this expression is $100\dot{x} - 10x\dot{x}$; which being made $= 0$, and divided by $10\dot{x}$, the equation is $10 - x = 0$; and hence $x = 10$. That is, the value of x is 10, when the expression $100x - 5x^2 \pm c$ is the greatest. As is easily tried: for if 10 be substituted for x in that expression, it becomes $\pm c + 500$: but if, for x, there be substituted any other number, whether greater or less than 10, that expression will always be found to be less than $\pm c + 500$, which is therefore its greatest possible value, or its maximum.

58. It is evident, that if a maximum or minimum be any way compounded with, or operated on, by a given constant quantity, the result will still be a maximum or minimum. That is, *if a maximum or minimum be increased, or decreased, or multiplied, or divided, by a given quantity, or any given power or root of it be taken;* the result will still be a maximum or minimum. Thus, if x be a maximum or minimum, then also is $x + a$, or $x - a$, or ax, or $\frac{x}{a}$, or x^n, or $\sqrt[n]{x}$, still a maximum or minimum. Also, *the logarithm of the same will be a maximum or a minimum.* And therefore, *if any proposed maximum or minimum can be made simpler by performing any of these operations, it is better to do so,* before the expression is put into fluxions.

59. When the expression for a maximum or minimum contains several variable letters or quantities; take the fluxion of it as often as there are variable letters; supposing first one

of them only to flow, and the rest to be constant; then another only to flow, and the rest constant; and so on for all of them: then putting each of these fluxions $= 0$, there will be as many equations as unknown letters, from which these may be all determined. For the fluxion of the expression must be equal to nothing in each of these cases; otherwise the expression might become greater or less, without altering the values of the other letters, which are considered as constant.

So, if it be required to find the values of x and y, when $4x^2 - xy + 2y$ is a minimum. Then we have,

First, - $8x\dot{x} - \dot{x}y = 0$, and $8x - y = 0$, or $y = 8x$.
Secondly, $2\dot{y} - x\dot{y} = 0$, and $2 - x = 0$, or $x = 2$.
And hence y or $8x = 16$.

60. *To find whether a proposed quantity admits of a Maximum or a Minimum.*

Every algebraic expression does not admit of a maximum or minimum, properly so called; for it may either increase continually to infinity, or decrease continually to nothing; and in both these cases there is neither a proper maximum nor minimum; for the true maximum is that finite value to which an expression increases, and after which it decreases again: and the minimum is that finite value to which the expression decreases, and after that it increases again. Therefore, when the expression admits of a maximum, its fluxion is positive before the point, and negative after it; but when it admits of a minimum, its fluxion is negative before, and positive after it. Hence then, taking the fluxion of the expression a little before the fluxion is equal to nothing, and again a little after the same; if the former fluxion be positive, and the latter negative, the middle state is a maximum; but if the former fluxion be negative, and the latter positive, the middle state is a minimum.

So, if we would find the quantity $ax - x^2$ a maximum or minimum; make its fluxion equal to nothing, that is, - - $a\dot{x} - 2x\dot{x} = 0$, or $(a - 2x)\dot{x} = 0$; dividing by $\dot{x}$, gives $a - 2x = 0$, or $x = \frac{1}{2}a$ at that state. Now, if in the fluxion $(a - 2x)\dot{x}$, the value of x be taken rather less than its true value, $\frac{1}{2}a$, that fluxion will evidently be positive; but if x be taken somewhat greater than $\frac{1}{2}a$ the value of $a - 2x$, and consequently of the fluxion, is as evidently negative. Therefore, the fluxion of $ax - x^2$ being positive before, and negative after the state when its fluxion is $= 0$, it follows that at this state the expression is not a minimum, but a maximum.

Again, taking the expression $x^3 - ax^2$, its fluxion $3x^2\dot{x} - 2ax\dot{x} = (3x - 2a)x\dot{x} = 0$; this divided by $x\dot{x}$ gives $3x - 2a = 0$, and $x = \frac{2}{3}a$, its true value when the fluxion of $x^3 - ax^2$ is equal to nothing. But now to know whether the given expression be a maximum or a minimum at that time, take x a little less than $\frac{2}{3}a$ in the value of the fluxion $(3x - 2a)\ x\dot{x}$, and this will evidently be negative; and again, taking x a little more than $\frac{2}{3}a$, the value of $3x - 2a$, or of the fluxion, is as evidently positive. Therefore the fluxion of $x^3 - ax^2$ being negative before that fluxion is $= 0$, and positive after it, it follows that in this state the quantity $x^3 - ax^2$ admits of a minimum, but not of a maximum.

SOME EXAMPLES FOR PRACTICE.

EXAM. 1. Of all triangles, ACB, constructed on the same base AB, and having the same perimeter, to determine that whose area or surface is the greatest.

Let p denote the semiperimeter, b the base AB, x the side AC, then BC will $= 2p - b - x$. Therefore putting s for the surface, we have by rule 3 for the area of triangles (pa. 31.)

$$s^2 = p\ (p - b)\ (p - x)\ (b + x - p).$$

Expressing this equation logarithmically, we have, 2 log. $s = \log. p + \log. (p - b) + \log. (p - x) + \log. (b + x - p)$ which (art. 58) is to be a max. or when put into fluxions equal to *zero* or nothing.

Hence $\dfrac{2\dot{s}}{s} = \dfrac{-\dot{x}}{p - x} + \dfrac{\dot{x}}{b + x - p}$;

or, dividing by $2\dot{x}$, and multiplying by s,

$$\frac{\dot{s}}{\dot{x}} = \frac{s}{2}\left(\frac{1}{b + x - p} - \frac{1}{p - x}\right) = 0.$$

Now, here it is evident, since s must be a max. that $\dfrac{s}{2}$ cannot $= 0$; consequently the second factor must: that is,

$$\frac{1}{b + x - p} - \frac{1}{p - x} = 0, \text{ or } b + x - p = p - x.$$

Therefore, $2p - b - x = x$, or AC $=$ BC; that is, the triangle must be isosceles.

Cor. Hence it follows that of all *isoperimetrical* triangles, the one which has the greatest surface is equilateral. A truth, indeed, which may be readily shown by a direct investigation.

EXAM. 2. Amongst all parallelopipedons of given magnitude, whose planes are respectively perpendicular to one another, to determine that which has the least surface.

Let x, y, and z, be the measures of the three edges of the required parallelopipedon. Then, since the magnitude is given,

$$\text{we have } xyz = a, \text{ a given magnitude;}$$
$$\text{and } 2xy + 2xz + 2yz = \text{a minimum.}$$

Here, substituting for z, and dividing by 2, there results

$$xy + x \cdot \frac{a}{xy} + y \cdot \frac{a}{xy} = \text{a min.}$$

$$\text{or, } u = xy + \frac{a}{y} + \frac{a}{x} = \text{a min.}$$

Therefore, adopting the principle of art. 59,

$$\left.\begin{aligned} \frac{\dot{u}}{\dot{x}} &= y - \frac{a}{x^2} = 0 \\ \text{and } \frac{\dot{u}}{\dot{y}} &= x - \frac{a}{y^2} = 0 \end{aligned}\right\} \text{must both obtain.}$$

$$\text{Hence, } y = \frac{a}{x^2} = a \div \left(\frac{a}{y^2}\right)^2 = a \cdot \frac{y^4}{a^2} = \frac{y^4}{a}.$$

$$\left.\begin{aligned} \text{Consequently } y &= a^{\frac{1}{3}} \\ x &= a^{\frac{1}{3}} \\ \therefore z &= a^{\frac{1}{3}} \end{aligned}\right\} \text{and thus it appears that the required parallelopipedon is a } \textit{cube}.$$

EXAM. 3 Divide a given arc A into two such parts, that the mth power of the sine of one part, multiplied into the nth power of the sine of the other part, shall be a maximum.

Let x and y be the parts: then $x + y = \text{A}$, and $\sin.^m x \times \sin.^n y = \text{a max.}$

In logs. $m \log. \sin. x + n \log. \sin. y = \text{a max.}$

$$\text{Hence, (art. 58) } \frac{m\dot{x} \cos. x}{\sin. x} + \frac{n\dot{y} \cos. y}{\sin. y} = 0.$$

$$\text{But } \dot{y} = -\dot{x} \therefore \frac{m\dot{x} \cos. x}{\sin. x} - \frac{n\dot{x} \cos. y}{\sin. y} = 0.$$

Hence $m \cot. x = n \cot. y$, or $m \tan. y = n \tan. x$,

$$\therefore \frac{m}{n} = \frac{\tan. x}{\tan. y} \ldots \text{and } \frac{m+n}{m-n} = \frac{\tan. x + \tan. y}{\tan. x - \tan. y} = \frac{\sin. (x+y)}{\sin. (x-y)}.$$

(See equa. 9 and 10, p. 18).

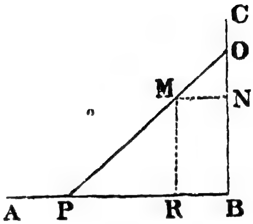
Hence x and y become known: and the same principle is

evidently applicable to three or more arcs, making together a given arc.

EXAM. 4. To find the longest straight pole that can be put up a chimney, whose height RM $= a$, from the floor to the mantel, and depth MN $= b$, from front to back, are given.

Here the longest pole that can be put up the chimney is, in fact, the *shortest* line PMO, which can be drawn through M, and terminated by BA and BC.

$$\left.\begin{array}{l}\text{Let } x = \text{sine}\\ \quad\; y = \text{cos.}\end{array}\right\}\text{ of OPB,}$$



$$x : \text{MR} \;(=a) :: 1 : \frac{a}{x} = \text{PM}$$

$$y : \text{MN} \;(=b) :: 1 : \frac{b}{y} = \text{MO}$$

$$\frac{a}{x} + \frac{b}{y} = \text{a min.}$$

$$\text{In flux.} -\frac{a\dot{x}}{x^2} - \frac{b\dot{y}}{y^2} = 0.$$

$$\text{But } x^2 + y^2 = 1, \therefore 2x\dot{x} = -2y\dot{y},$$

$$\text{and } -\dot{y} = \frac{x\dot{x}}{y}.$$

Substituting this for $\dot{y}$ above, it becomes

$$-\frac{a\dot{x}}{x^2} + \frac{bx\dot{x}}{y^3} = 0,$$

$$\therefore bx^3 = ay^3$$

$$\frac{x^3}{y^3} = \frac{a}{b},$$

$$\frac{x}{y} = \tan.\, \text{P} = \sqrt[3]{\frac{a}{b}}$$

$$\text{PO} = a \operatorname{cosec} \text{P} + b \sec. \text{P} = a\sqrt{1 + \frac{b^{\frac{2}{3}}}{a^{\frac{2}{3}}}} + b\sqrt{1 + \frac{a^{\frac{2}{3}}}{b^{\frac{2}{3}}}}.$$

EXAM. 5. To divide a line, or any other given quantity a, into two parts, so that their rectangle or product may be the greatest possible.

EXAM. 6. To divide the given quantity a into two parts such, that the product of the m power of one, by the n power of the other, may be a maximum.

Exam. 7. To divide the given quantity a into three parts such, that the continual product of them all may be a maximum.

Exam. 8. To divide the given quantity a into three parts such, that the continual product of the 1st, the square of the 2d, and the cube of the 3d, may be a maximum.

Exam. 9. To determine a fraction such, that the difference between its m power and n power shall be the greatest possible.

Exam. 10. To divide the number 80 into two such parts, x and y, that $2x^2 + xy + 3y^2$ may be a minimum.

Exam. 11. To find the greatest rectangle that can be inscribed in a given right-angled triangle.

Exam. 12. To find the greatest rectangle that can be inscribed in the quadrant of a given circle.

Exam. 13. To find the least right-angled triangle that can circumscribe the quadrant of a given circle.

Exam. 14. To find the greatest rectangle inscribed in, and the least isosceles triangle circumscribed about, a given semi-ellipse.

Exam. 15. To determine the same for a given parabola.

Exam. 16. To determine the same for a given hyperbola.

Exam. 17. To inscribe the greatest cylinder in a given cone; or to cut the greatest cylinder out of a given cone.

Exam. 18. To determine the dimensions of a rectangular cistern, capable of containing a given quantity a of water, so as to be lined with lead at the least possible expense.

Exam. 19. Required the dimensions of a cylindrical tankard, to hold one quart of ale measure, that can be made of the least possible quantity of silver, of a given thickness.

Exam. 20. To cut the greatest parabola from a given cone.

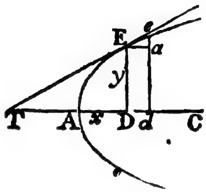
Exam. 21. To cut the greatest ellipse from a given cone.

Exam. 22. To find the value of x when x^x is a minimum.

For various examples of the maxima and minima of mechanical quantities, see vol. iii.

THE METHOD OF TANGENTS; OR OF DRAWING TANGENTS TO CURVES.

61. THE Method of Tangents, is a method of determining the quantity of the tangent and subtangent of any algebraic curve; the equation of the curve being given. Or, *vice versa*, the nature of the curve, from the tangent given.



If AE be any curve, and E be any point in it, to which it is required to draw a tangent TE. Draw the ordinate ED: then if we can determine the subtangent TD, limited between the ordinate and tangent, in the axis produced, by joining the points T, E, the line TE will be the tangent sought.

62. Let *dae* be another ordinate, indefinitely near to DE, meeting the curve, or tangent produced in *e*; and let E*a* be parallel to the axis AD. Then is the elementary triangle E*ea* similar to the triangle TDE; and

therefore - $ea : a\text{E} :: \text{ED} : \text{DT}$.

But - - $ea : a\text{E} ::$ flux. ED : flux. AD.

Therefore - flux. ED : flux. AD :: DE : DT.

That is, - $\dot{y} : \dot{x} :: y : \dfrac{y\dot{x}}{\dot{y}} = \text{DT}$;

which is therefore the general value of the subtangent sought; where x is the absciss AD, and y the ordinate DE.

Hence we have this general rule.

GENERAL RULE.

63. By means of the given equation of the curve, when put into fluxions, find the value of either $\dot{x}$ or $\dot{y}$, or of $\dfrac{\dot{x}}{\dot{y}}$ which value substitute for it in the expression $\text{DT} = \dfrac{y\dot{x}}{\dot{y}}$, and, when reduced to its simplest terms, it will be the value of the subtangent sought.

EXAMPLES.

EXAM. 1. Let the proposed curve be that which is defined, or expressed, by the equation $ax^2 + xy^2 - y^3 = 0$.

Here the fluxion of the equation of the curve is
$2ax\dot{x} + y^2\dot{x} + 2xy\dot{y} - 3y^2\dot{y} = 0$; then, by transposition,
$2ax\dot{x} + y^2\dot{x} = 3y^2\dot{y} - 2xy\dot{y}$; and hence, by division,
$\frac{\dot{x}}{\dot{y}} = \frac{3y^2 - 2xy}{2ax + y^2}$; consequently $\frac{y\dot{x}}{\dot{y}} = \frac{3y^3 - 2xy^2}{2ax + y^2}$,
which is the value of the subtangent TD sought.

EXAM. 2. To draw a tangent to a circle; the equation of which is $ax - x^2 = y^2$; where x is the absciss, y the ordinate, and a the diameter.

EXAM. 3. To draw a tangent to a parabola; its equation being $px = y^2$; where p denotes the parameter of the axis.

EXAM. 4. To draw a tangent to an ellipse; its equation being $c^2(ax - x^2) = a^2y^2$; where a and c are the two axes.

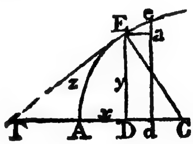
EXAM. 5. To draw a tangent to an hyperbola; its equation being $c^2(ax + x^2) = a^2y^2$; where a and c are the two axes.

EXAM. 6. To draw a tangent to the hyperbola referred to the asymptote as an axis; its equation being $xy = a^2$; where a^2 denotes the rectangle of the absciss and ordinate answering to the vertex of the curve.

By slight and obvious extensions of the same principles, tangents may be drawn to spirals, and asymptotes may be drawn to such curves as admit of them.

OF RECTIFICATIONS; OR, TO FIND THE LENGTHS OF CURVE LINES.

64. RECTIFICATION, is the finding the length of a curve line, or finding a right line equal to a proposed curve.



By art. 10 it appears, that the elementary triangle Eae, formed by the increments of the absciss, ordinate, and curve, is a right-angled triangle, of which the increment of the curve is the hypothenuse; and therefore the square of the latter is equal to the sum of the squares of the two former; that is, Ee^2 = E$a^2 + ae^2$. Or, substituting, for the increments, their proportional fluxions, it is $\dot{z}\dot{z} = \dot{x}\dot{x} + \dot{y}\dot{y}$, or $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; where z denotes any curve line AE, x its absciss AD, and y its ordinate DE. Hence this rule.

RULE.

65. From the given equation of the curve put into fluxions, find the value of $\dot{x}^2$ or $\dot{y}^2$, which value substitute instead of it in the equation $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; then the fluents, being taken, will give the value of z, or the length of the curve, in terms of the absciss or ordinate.

EXAMPLES.

EXAM. 1. To find the length of the arc of a circle, in terms both of the sine, versed sine, tangent, and secant.

The equation of the circle may be expressed in terms of the radius, and either the sine, or the versed sine, or tangent, or secant, &c. of an arc. Let therefore the radius of the circle be CA or CE $= r$, the versed sine AD (of the arc AE) $= x$, the right sine DE $= y$, the tangent TE $= t$, and the secant CT $= s$; then, by the nature of the circle, there arise these equations, viz.

$$y^2 = 2rx - x^2 = \frac{r^2t^2}{r^2 + t^2} = \frac{s^2 - r^2}{s^2} r^2.$$

Then, by means of the fluxions of these equations, with the general fluxional equation $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$, are obtained the following fluxional forms, for the fluxion of the curve; the fluent of any one of which will be the curve itself; viz.

$$\dot{z} = \frac{r\dot{x}}{\sqrt{2rx - xx}} = \frac{r\dot{y}}{\sqrt{r^2 - y^2}} = \frac{r^2\dot{t}}{r^2 + t^2} = \frac{r^2\dot{s}}{\sqrt{s^2 - r^2}} *.$$

* These formulæ are, obviously, analogous to those given in art. 30, p. 303, and are so many forms of fluxions whose fluents become known. Thus the fluent of an expression, such as $\frac{r\dot{x}}{\sqrt{(2rx - x^2)}}$, is a circular arc whose radius is $= r$ and versed sine $= x$. The fluent of an expression such as $\frac{r^2\dot{t}}{r^2 + t^2}$ is a circular arc whose radius is $= r$ and tangent $= t$: and so of the rest.

Conversely, the same formulæ, or those just referred to, serve to assign the relative magnitudes of the *differences* in any parts of a table of natural sines, of natural tangents, &c. Thus, $\dot{t} = \frac{r^2 + t^2}{r^2}\dot{z} = \dot{z} \times \text{sec.}^2$ of arc to tan. t, consequently, the

Hence the value of the curve, from the fluent of each of these, expressed in series, gives the four following forms, in series, viz. putting $d = 2r$ the diameter, the curve is

$$z = (1 + \frac{x}{2.3d} + \frac{3x^2}{2.4.5d^2} + \frac{3.5x^3}{2.4.6.7d^3} + \&c.)\sqrt{dx},$$

$$= (1 + \frac{y^2}{2.3r^2} + \frac{3y^4}{2.4.5r^4} + \frac{3.5y^6}{2.4.6.7r^6} + \&c.)\,y,$$

$$= (1 - \frac{t^2}{3r^2} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \frac{t^8}{9r^8} - \&c.)\,t,$$

$$= (\frac{s-r}{s} + \frac{s^3-r^3}{2.3s^3} + \frac{3(s^5-r^5)}{2.4.5s^5} + \&c.)\,r.$$

Now, it is evident, that the simplest of these series, is the third in order, or that which is expressed in terms of the tangent. That form will therefore be the fittest to calculate an example by in numbers. And for this purpose it will be convenient to assume some arc whose tangent, or at least the square of it, is known to be some small simple number. Now, the arc of 45 degrees, it is known, has its tangent equal to the radius; and therefore, taking the radius $r = 1$, and consequently the tangent of $45°$, or t, $= 1$ also, in this case the arc of $45°$ to the radius 1, or the arc of the quadrant to the diameter 1, will be equal to the infinite series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.$

But as this series converges very slowly, it will be proper to take some smaller arc, that the series may converge faster; such as the arc of 30 degrees, the tangent of which is $= \sqrt{\frac{1}{3}}$, or its square $t^2 = \frac{1}{3}$: which being substituted in the series, the length of the arc of $30°$ comes out - - -

$(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \frac{1}{9.3^4} - \&c.)\sqrt{\frac{1}{3}}$. Hence, to compute these terms in decimal numbers, after the first, the succeeding terms will be found by dividing, always by 3, and these quotients again by the absolute numbers 3, 5, 7, 9, &c.; and lastly, adding every other term together, into two sums, the one the sum of the positive terms, and the other the sum of the negative ones; then lastly, the one sum taken from the other, leaves the length of the arc of 30 degrees; which being the 12th part of the whole circumference when

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tabular differences of the tangents vary as the squares of the secants. Hence, those differences, at $0°$, at $45°$, and at $60°$, are as 1^2, $(\sqrt{2})^2$, and 2^2, or as 1, 2, and 4. This suggests an application of these formulæ which will often be found useful.

the radius is 1, or the 6th part when the diameter is 1, consequently 6 times that arc will be the length of the whole circumference to the diameter 1. Therefore, multiplying the first term $\sqrt{\frac{1}{3}}$ by 6, the product is $\sqrt{12} = 3{\cdot}4641016$; and hence the operation, true to 7 places of decimals, will be conveniently made as follows:

		+ Terms.	− Terms.
1)	3·4641016 (	3·4641016	
3)	1·1547005 (		0·3849002
5)	3849002 (	769800	
7)	1283001 (		183286
9)	427667 (	47519	
11)	142556 (		12960
13)	47519 (	3655	
15)	15840 (		1056
17)	5280 (	311	
19)	1760 (		93
21)	587 (	28	
23)	196 (		8
25)	65 (	3	
27)	22 (		1
		+3·5462332	−0·4046406
		−0·4046406	

So that at last 3·1415926 is the whole circumference to the diameter 1*.

Exam. 2. To find the length of a parabola.

Exam. 3. To find the length of the semicubical parabola, whose equation is $ax^2 = y^3$.

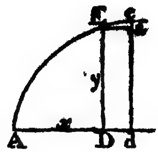
Exam. 4. To find the length of an elliptical curve.

Exam. 5. To find the length of an hyperbolic curve.

OF QUADRATURES; OR, FINDING THE AREAS OF CURVES.

66. The Quadrature of Curves, is the measuring their areas, or finding a square, or other right-lined space, equal to a proposed curvilineal one.

* For this value, true to 100 places of decimals; and indeed for many curious and important investigations in reference to rectifications, quadratures, &c. see *Hutton's Mensuration*.



By art. 9. it appears, that any flowing quantity being drawn into the fluxion of the line along which it flows, or in the direction of its motion, there is produced the fluxion of the quantity generated by the flowing. That is, DE × Dd or $y\dot{x}$ is the fluxion of the area ADE. Hence this rule.

RULE.

67. From the given equation of the curve, find the value either of $\dot{x}$ or of y; which value substitute instead of it in the expression $y\dot{x}$; then the fluent of that expression, being taken, will be the area of the curve sought.

EXAMPLES.

EXAM. 1. To find the area of the common parabola.

The equation of the parabola being $ax = y^2$; where a is the parameter, x the absciss AD, or part of the axis, and y the ordinate DE.

From the equation of the curve is found $y = \sqrt{ax}$. This substituted in the general fluxion of the area $y\dot{x}$ gives $\dot{x}\sqrt{ax}$ or $a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$ the fluxion of the parabolic area; and the fluent of this, or $\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} = \frac{2}{3}x\sqrt{ax} = \frac{2}{3}xy$, is the area of the parabola ADE, which is therefore equal to $\frac{2}{3}$ of its circumscribing rectangle.

EXAM. 2. To square the circle, or find its area.

The equation of the circle being $y^2 = ax - x^2$, or $y = \sqrt{ax - x^2}$, where a is the diameter; by substitution, the general fluxion of the area $y\dot{x}$, becomes $\dot{x}\sqrt{ax - x^2}$, for the fluxion of the circular area. But as the fluent of this cannot be found in finite terms, the quantity $\sqrt{ax - x^2}$ is thrown into a series, by extracting the root, and then the fluxion of the area becomes

$$\dot{x}\sqrt{ax} \times \left(1 - \frac{x}{2a} - \frac{x^2}{2.4a^2} - \frac{1.3x^3}{2.4.6a^3} - \frac{1.3.5x^4}{2.4.6.8a^4} - \&c.\right);$$

and then the fluent of every term being taken, it gives

$$x\sqrt{ax} \times \left(\frac{2}{3} - \frac{1.x}{5a} - \frac{1.x^2}{4.7a^2} - \frac{1.3x^3}{4.6.9a^3} - \frac{1.3.5x^4}{4.6.8.11a^4} - \&c\right);$$

for the general expression of the semisegment ADE.

And when the point D arrives at the extremity of the dia-

meter, then the space becomes a semicircle, and $x = a$; and then the series above becomes barely

$$a^2\left(\frac{2}{3} - \frac{1}{5} - \frac{1}{4.7} - \frac{1.3}{4.6.9} - \frac{1.3.5}{4.6.8.11} - \&c.\right)$$

for the area of the semicircle whose diameter is a.

If, instead of taking the equation of the circle having the origin of the co-ordinates at the circumference, the equation $x^2 + y^2 = r^2$ be taken, regarding the origin of the co-ordinates at the centre; and if, still farther, r be taken $= 1$, then $y = \sqrt{1 - x^2} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \&c.$ Taking this value of y for it, in the expression $y\dot{x}$, the correct fluent will be $x - \frac{x^3}{6} - \frac{x^5}{40} - \frac{x^7}{112} - \frac{5x^9}{1152} - \frac{7x^{11}}{2816}$, for the area of the portion CFBD (fig. pa. 2.) Now, if arc BD $= 30^0$, then CF $= x = \frac{1}{2}$, and the sum of the series $= \cdot 4783057$. From which deducting the area of the triangle CFB $= \frac{1}{4} \cdot \frac{1}{2}\sqrt{3} = \cdot 2165063$, there remains $\cdot 2617994$ for the area of the sector CBD. Twelve times this, or $3 \cdot 1415928$ &c. expresses the area of the circle whose diameter is 2.

EXAM. 3. To find the area of any parabola, whose equation is $a^m x^n = y^{m+n}$.

EXAM. 4. To find the area of an ellipse.

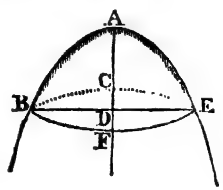
EXAM. 5. To find the area of an hyperbola.

EXAM. 6. To find the area between the curve and asymptote of an hyperbola.

EXAM. 7. To find the like area in any other hyperbola whose general equation is $x^m y^n = a^{m+n}$.

TO FIND THE SURFACES OF SOLIDS.

68. In the solid formed by the rotation of any curve about its axis, the surface may be considered as generated by the circumference of an expanding circle, moving perpendicularly along the axis, but the expanding circumference moving along the arc or curve of the solid. Therefore, as the fluxion of any generated quantity, is produced by drawing the generating quantity into the fluxion of the line or direction in which it moves, the fluxion of the surface will be found by drawing the circumference of the generating circle into the fluxion of the curve. That is, the fluxion of the surface BAE, is equal to AE drawn into the circumference BCEF, whose radius is the ordinate DE.



69. But if π be $= 3.141593$, the circumference of a circle whose diameter is 1, $x =$ AD the absciss, $y =$ DE the ordinate, and $z =$ AE the curve; then $2y =$ the diameter BE, and $2\pi y =$ the circumference BCEF; also, AE $= \dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$: therefore $2\pi y\dot{z}$ or $2\pi y\sqrt{\dot{x}^2 + \dot{y}^2}$ is the fluxion of the surface. And consequently if, from the given equation of the curve, the value of $\dot{x}$ or $\dot{y}$ be found, and substituted in this expression $2\pi y\sqrt{\dot{x}^2 + \dot{y}^2}$, the fluent of the expression, being then taken, will be the surface of the solid required.

EXAMPLES.

EXAM. 1. To find the surface of a sphere, or of any segment.

In this case, AE is a circular arc, whose equation is $y^2 = ax - x^2$, or $y = \sqrt{(ax - x^2)}$.

The fluxion of this gives $\dot{y} = \frac{a - 2x}{2\sqrt{ax - x^2}} \cdot \dot{x} = \frac{a - 2x}{2y}\dot{x}$;

hence $\dot{y}^2 = \frac{a^2 - 4ax + 4x^2}{4y^2}\dot{x}^2 = \frac{a^2 - 4y^2}{4y^2}\dot{x}^2$; consequently

$\dot{x}^2 + \dot{y}^2 = \frac{a^2\dot{x}^2}{4y^2}$, and $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{a\dot{x}}{2y}$.

This value of $\dot{z}$, the fluxion of a circular arc, may be found more easily thus: In the fig. to art. 64, the two triangles EDC, Eae are equiangular, being each of them equiangular to the triangle ETC : conseq. ED : EC : : Ea : Ee, that is,

$y : \frac{1}{2}a :: \dot{x} : \dot{z} = \frac{a\dot{x}}{2y}$, the same as before.

The value of $\dot{z}$ being found, by substitution is obtained $2\pi y\dot{z} = a\pi\dot{x}$ for the fluxion of the spherical surface, generated by the circular arc in revolving about the diameter AD. And the fluent of this gives acx for the said surface of the spherical segment BAE.

But ac is equal to the whole circumference of the generating circle; and therefore it follows, that the surface of any spherical segment, is equal to the same circumference of the generating circle, drawn into x or AD, the height of the segment.

Also when x or AD becomes equal to the whole diameter a, the expression $a\pi x$ becomes $a\pi a$ or πa^2, or 4 times the area of the generating circle, for the surface of the whole sphere.

And these agree with the rules before found in Mensuration of Solids.

EXAM. 2. To find the surface of a spheroid.

EXAM. 3. To find the surface of a paraboloid.

EXAM. 4. To find the surface of an hyperboloid.

TO FIND THE CONTENTS OF SOLIDS.

70. Any solid which is formed by the revolution of a curve about its axis (see last fig.), may also be conceived to be generated by the motion of the plane of an expanding circle, moving perpendicularly along the axis. And therefore the area of that circle being drawn into the fluxion of the axis will produce the fluxion of the solid. That is, $\dot{\text{AD}} \times$ area of the circle BCF, whose radius is DE, or diameter BE, is the fluxion of the solid, by art. 9.

71. Hence, if AD $= x$, DE $= y$, $\pi = 3\cdot141593$; because πy^2 is equal to the area of the circle BCF; therefore $cy^2\dot{x}$ is the fluxion of the solid. Consequently if, from the given equation of the curve, the value of either y^2 or x be found, and that value substituted for it in the expression $\pi y^2\dot{x}$, the fluent of the resulting quantity, being taken, will be the solidity of the figure proposed.

EXAMPLES.

Exam. 1. To find the solidity of a sphere, or any segment.

The equation to the generating circle being $y^2 = ax - x^2$, where a denotes the diameter, by substitution, the general fluxion of the solid $\pi y^2\dot{x}$, becomes $\pi ax\dot{x} - \pi x^2\dot{x}$, the fluent of which gives $\frac{1}{2}\pi ax^2 - \frac{1}{3}\pi x^3$, or $\frac{1}{6}\pi x^2(3a - 2x)$, for the solid content of the spherical segment BAE, whose height AD is x.

When the segment becomes equal to the whole sphere, then $x = a$, and the above expression for the solidity, becomes $\frac{1}{6}\pi a^3$ for the solid content of the whole sphere.

And these deductions agree with the rules before given and demonstrated in the Mensuration of Solids.

Exam. 2. To find the solidity of a spheroid.

Exam. 3. To find the solidity of a paraboloid.

Exam. 4. To find the solidity of an hyperboloid.

TO FIND LOGARITHMS.

72. It has been proved, art. 23, that the fluxion of the hyperbolic logarithm of a quantity, is equal to the fluxion of the quantity divided by the same quantity. Therefore, when any quantity is proposed, to find its logarithm; take the fluxion of that quantity, and divide it by the same quantity;

then take the fluent of the quotient, either in a series or otherwise, and it will be the logarithm sought; when corrected as usual, if need be; that is, the hyperbolic logarithm.

73. But, for any other logarithm, multiply the hyperbolic logarithm, above found, by the modulus of the system, for the logarithm sought.

Note. The modulus of the hyperbolic logarithms, is 1; and the modulus of the common logarithms, is ·43429448190 &c; and, in general, the modulus of any system, is equal to the logarithm of 10 in that system divided by the number 2·3025850929940 &c, which is the hyp. log. of 10. Also, the hyp. log. of any number, is in proportion to the com. log. of the same number, as unity or 1 is to ·43429 &c, or as the number 2·302585 &c, is to 1; and therefore, if the common log. of any number be multiplied by 2·302585 &c, it will give the hyp. log. of the same number; or if the hyp. log. be divided by 2·302585 &c, or multiplied by ·43429 &c, it will give the common logarithm. See farther, pa. 125.

Exam. 1. To find the log. of $\frac{a+x}{a}$.

Denoting any proposed number z, whose logarithm is required to be found, by the compound expression - - - $\frac{a+x}{a}$, the fluxion of the number $\dot{z}$, is $\frac{\dot{x}}{a}$, and the fluxion of the log. $\frac{\dot{z}}{z} = \frac{\dot{x}}{a+x} = \frac{\dot{x}}{a} - \frac{x\dot{x}}{a^2} + \frac{x^2\dot{x}}{a^3} - \frac{x^3\dot{x}}{a^4} +$ &c.

Then the fluent of these terms gives the logarithm of z or logarithm of $\frac{a+x}{a} = \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4}$ &c.

Writing $-x$ for x, gives log. $\frac{a-x}{a} = -\frac{x}{a} - \frac{x^2}{2a^2} - \frac{x^3}{3a^3} - \frac{x^4}{4a^4}$ &c.

Div. these numb. and subtr. their logs. gives $\Big\}$ log. $\frac{a+x}{a-x} = \frac{2x}{a} + \frac{2x^3}{3a^3} + \frac{2x^5}{5a^5}$ &c.

Also, because $\frac{a}{a \pm x} = 1 \div \frac{a \pm x}{a}$, or log. $\frac{a}{a \pm x} = 0 - \log. \frac{a \pm x}{a}$;

therefore log. of $\frac{a}{a+x}$ is $-\frac{x}{a} + \frac{x^2}{2a^2} - \frac{x^3}{3a^3} + \frac{x^4}{4a^4}$ &c,

and the log. of $\frac{a}{a-x}$ is $+\frac{x}{a} + \frac{x^2}{2a^2} + \frac{x^3}{3a^3} + \frac{x^4}{4a^4}$ &c.

the prod. gives log. $\frac{a^2}{a^2 - x^2} = \frac{x^2}{a^2} + \frac{x^4}{2a^4} + \frac{x^6}{3a^6} +$ &c.

Now, for an example in numbers, suppose it were required to compute the common logarithm of the number 2. This will be best done by the series.

$$\log.\text{ of }\frac{a+x}{a-x}=2\text{M}\times(\frac{x}{a}+\frac{x^3}{3a^3}+\frac{x^5}{5a^5}+\frac{x^7}{7a^7}\text{ \&c.})$$

Making $\frac{a+x}{a-x}=2$, gives $a=3x$; conseq. $\frac{x}{a}=\frac{1}{3}$, and $\frac{x^2}{a^2}=\frac{1}{9}$, which is the constant factor for every succeeding term; also, $2\text{M}=2\times{\cdot}43429448190={\cdot}868588964$; therefore the calculation will be conveniently made, by first dividing this number by 3, then the quotients successively by 9, and lastly these quotients in order by the respective numbers 1, 3, 5, 7, 9, &c, and after that, adding all the terms together, as follows:

3)	·868588964				
9)	289529654	1)	·289529654	(	·289529654
9)	32169962	3)	32169962	(	10723321
9)	3574440	5)	3574440	(	714888
9)	397160	7)	397160	(	56737
9)	44129	9)	44129	(	4903
9)	4903	11)	4903	(	446
9)	545	13)	545	(	42
9)	61	15)	61	(	4

Sum of the terms gives log. 2 = ·301029995

EXAM. 2. To find the log. of $\frac{a+x}{b}$.

EXAM. 3. To find the log. of a^2-x.

EXAM. 4. To find the log. of 3.

EXAM. 5. To find the log. of 5.

EXAM. 6. To find the log. of 11.

Note. The hyp. log. of $1+x$, is $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+$ &c.

And the nat. numb. to the hyp. log. y, is thus expressed,

$$1+y+\tfrac{1}{2}y^2+\frac{1}{2\cdot3}y^3+\frac{1}{2.3.4}y^4+\frac{1}{2.3.4.5}y^5+\text{\&c.}$$

74. The above given series $\log.\frac{a+x}{a-x}=2\text{M}\left(\frac{x}{a}+\frac{x^3}{3a^3}+\frac{x^5}{5a^5}+\text{\&c.}\right)$ when $a=1$, becomes $\log.\frac{1+x}{1-x}=2\text{M}(x+\frac{1}{3}x^3+\frac{1}{5}x^5+\text{\&c.})$: and this latter, by substituting

z for $\frac{1+x}{1-x}$, and consequently $\frac{z-1}{z+1}$ for x, becomes

$$\log. z = 2\text{M}\left\{\frac{z-1}{z+1} + \tfrac{1}{3}\left(\frac{z-1}{z+1}\right)^3 + \tfrac{1}{5}\left(\frac{z-1}{z+1}\right)^5 + \&c.\right\}$$

a series of rapid convergence, and convenient application, when z is a small integer.

If, now, in this we suppose

$$z = \frac{n^2}{n^2-1} = \frac{n^2}{(n-1)(n+1)}$$

so that $\frac{z-1}{z+1} = \frac{1}{2n^2-1}$, then the formula becomes

$$\log. \frac{n^3}{(n-1)(n+1)} = 2\text{M}\left\{\frac{1}{2n^2-1} + \tfrac{1}{3}\left(\frac{1}{2n^2-1}\right)^3 + \tfrac{1}{5}\left(\frac{1}{(2n^2-1)}\right)^5 + \&c.\right\}$$

But $\log. \frac{n^2}{(n-1)(n+1)} = 2 \log. n - \log. (n-1) - \log. (n+1)$; therefore, putting N for the series

$$2\text{M}\left\{\frac{1}{2n^2-1} + \tfrac{1}{3}\left(\frac{1}{2n^2-1}\right)^3 + \tfrac{1}{5}\left(\frac{1}{2n^2-1}\right)^5 + \&c.\right\}$$

we have this formula,

$$2 \log. n - \log. (n-1) - \log. (n+1) = \text{N}:$$

and hence, as often as we have the logarithms of any two of three numbers whose common difference is unity, the logarithm of the remaining number may be found. Example. Having given

the common log. of $9 = 0{\cdot}95424250943$
the common log. of $10 = 1$;

it is required to find the common logarithm of 11.

Here we have $n = 10$, so that the formula gives in this case $2 \log. 10 - \log. 9 - \log. 11 = \text{N}$, and hence we have

$$\log. 11 = 2 \log. 10 - \log. 9 - \text{N},$$

$$\text{where N} = \frac{2\text{M}}{199} + \frac{2\text{M}}{3.199^3} + \&c.$$

M being $\cdot 43429448190$.

Calculation of N.

$$\text{A} = \frac{2\text{M}}{199} = \cdot 00436476866$$

$$\text{B} = \frac{\text{A}}{3.199^2} = \cdot 00000003674$$

$$\text{N} = \cdot 00436480540$$

$$2 \log. 10 = 2{\cdot}00000000000$$

$$\log. 9 = 0{\cdot}95424250943$$
$$\text{N} = 0{\cdot}00436480540$$

$$\log. 9 + \text{N} = 0{\cdot}95860731483 : \text{taken from}$$

2 log. 10, leaves log. 11 = 1·04139268517

75. Here the series expressed by N converges very fast, so that two of its terms are sufficient to give the logarithm true to 10 places of decimals. But the logarithm of 11 may be expressed by the logarithms of smaller numbers and a series which converges still more rapidly, by the following artifice, which will apply also to some other numbers. Because the numbers 98, 99, and 100 are the products of numbers, the greatest of which is 11, for $98 = 2 \times 7^2$, $99 = 9 \times 11$, and $100 = 10 \times 10$, it follows that if we have an equation composed of terms which are the logarithms of these three numbers, it may be resolved into another, the terms of which shall be the logarithms of the number 11 and other smaller numbers. Now by the preceding formula, if we put 99 for n, we have

$$2 \log. 99 - \log. 98 - \log. 100 = \text{N}.$$

that is, substituting log. 9 + log. 11 for log. 99, log. 2 + 2 log. 7 for log. 98, and 2 log. 10 for log. 100,

$$2 \log. 9 + 2 \log. 11 - \log. 2 - 2 \log. 7 - 2 \log. 10 = \text{N},$$

and hence by transposition, &c.

$$\log. 11 = \tfrac{1}{2}\text{N} + \tfrac{1}{2}\log. 2 + \log. 7 - \log. 9 + \log. 10;$$

and in this equation,

$$\text{N} = \frac{2\text{M}}{19601} + \frac{1}{3}\frac{2\text{M}}{19601^3} + \&\text{c}.$$

The first term alone of this series is sufficient to give the logarithm of 11 true to 14 places.

76. When it is required to find the logarithm of a high number, as for example 1231, we may proceed as follows:

$$\log. 1231 = \log. (1230 + 1) = \log. \left\{ 1230\left(1 + \frac{1}{1230}\right) \right\}$$
$$= \log. 1230 + \log. \left(1 + \frac{1}{1230}\right)$$

Again, log. 1230 = log. 2 + log. 5 + log. 123, and

$$\log. 123 = \log. \left\{ 120\left(1 + \frac{1}{40}\right) \right\}$$
$$= \log. 120 + \log. \left(1 + \frac{1}{40}\right)$$

$\log. 120 = \log. (2^3 \times 3 \times 5) = 3 \log. 2 + \log. 3 + \log. 5.$
Therefore

$$\log. 1231 = 4 \log. 2 + \log. 3 + 2 \log. 5 + \log. \left(1 + \frac{1}{40}\right) + \log. \left(1 + \frac{1}{1230}\right)$$

Thus the logarithm of the proposed number is expressed by the logarithms of 2, 3, 5, and the logarithms of $1 + \frac{1}{40}$, $1 + \frac{1}{1230}$, all of which may be easily found by the formulæ already delivered.

77. When it is required to interpose one logarithm between a series of equidistant terms in a table, it may be effected upon the principle of interpolation by means of the well-known theorem; viz.

$$a - nb + n \cdot \frac{n-1}{2} \cdot c - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} d + \&c. = 0.$$

Thus, suppose there were given the logs. of 101, 102, 104, and 105, and that of 103 were required.

Here the number of equal intervals is 4, and of terms 5; so that the general form becomes

$$a - 4b + 6c - 4d + e = 0; \text{ and } c = \tfrac{1}{6}[4(b + d) - (a + e)].$$

$$\begin{aligned} a &= 2{\cdot}0043214 \\ b &= 2{\cdot}0086002 \\ d &= 2{\cdot}0170333 \\ e &= 2{\cdot}0211893 \end{aligned}$$

$$\begin{aligned} 4(b + d) &= 16{\cdot}1025340 \\ a + e &= 4{\cdot}0255107 \\ \hline &6)12{\cdot}0770233 \\ \hline \text{Log. of } 103 &= 2{\cdot}0128372 \end{aligned}$$

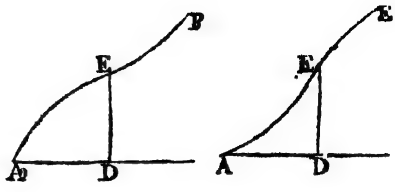
Exam. 1. Given the logs. of 999 and 1000. Required the log. of 1001.

Exam. 2. Given the logs. of 51, 53, 57, and 59; to find the log. of 55.

On this interesting and important subject, consult the Introduction to Dr. Hutton's Mathematical Tables, and Hellins's Mathematical Essays.

TO FIND THE POINTS OF INFLEXION, OR OF CONTRARY FLEXURE IN CURVES.

78. THE Point of Inflection in a curve is that point of it which separates the concave from the convex part, lying between the two; or where the curve changes from concave to convex, or from convex to concave, on the same side of the curve. Such as the point E in the annexed figures, where the former of the two is concave towards the axis AD, from A to E, but convex from E to F; and on the contrary, the latter figure is convex from A to E, and concave from E to F.



79. From the nature of curvature, as has been remarked before at art. 32, it is evident, that when a curve is concave towards an axis, then the fluxion of the ordinate decreases, or is in a decreasing ratio, with regard to the fluxion of the absciss; but, on the contrary, that it increases, or is in an increasing ratio to the fluxion of the absciss, when the curve is convex towards the axis; and consequently those two fluxions are in a constant ratio at the point of inflexion, where the curve is neither convex nor concave; that is, $\dot{x}$ is to $\dot{y}$ in a constant ratio, or $\frac{\dot{y}}{\dot{x}}$ or $\frac{\dot{x}}{\dot{y}}$ is a constant quantity. But constant quantities have no fluxion, or their fluxion is equal to nothing; so that, in this case, the fluxion of $\frac{\dot{y}}{\dot{x}}$ or of $\frac{\dot{x}}{\dot{y}}$ is equal to nothing. And hence we have this general rule:

80. Put the given equation of the curve into fluxions; from which find either $\frac{\dot{y}}{\dot{x}}$ or $\frac{\dot{x}}{\dot{y}}$. Then take the fluxion of this ratio, or fraction, and put it equal to 0 or nothing; and from this last equation find also the value of the same $\frac{\dot{x}}{\dot{y}}$ or $\frac{\dot{y}}{\dot{x}}$. Then put this latter value equal to the former, which

will form an equation; from which, and the first given equation of the curve, x and y will be determined, being the absciss and ordinate answering to the point of inflexion in the curve, as required.

EXAMPLES.

EXAM. 1. To find the point of inflexion in the curve whose equation is $ax^2 = a^2y + x^2y$.

This equation in fluxions is $2ax\dot{x} = a^2\dot{y} + 2xy\dot{x} + x^2\dot{y}$, which gives $\frac{\dot{x}}{\dot{y}} = \frac{a^2 + x^2}{2ax - 2xy}$. Then the fluxion of this quantity made $= 0$, gives $2x\dot{x}(ax - xy) = (a^2 + x^2) \times (a\dot{x} - \dot{x}y - x\dot{y})$; and this again gives $\frac{\dot{x}}{\dot{y}} = \frac{a^2 + x^2}{a^2 - x^2} \times \frac{x}{a - y}$.

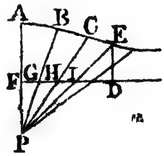
Lastly, this value of $\frac{\dot{x}}{\dot{y}}$ being put equal the former, gives $\frac{a^2 + x^2}{a^2 - x^2} \cdot \frac{x}{a - y} = \frac{a^2 + x^2}{2x} \cdot \frac{1}{a - y}$; and hence $2x^2 = a^2 - x^2$, or $3x^2 = a^2$, and $x = a\sqrt{\tfrac{1}{3}}$, the absciss.

Hence also, from the original equation, - - - - - -

$y = \frac{ax^2}{a^2 + x^2} = \frac{\frac{1}{3}a^3}{\frac{4}{3}a^2} = \frac{1}{4}a$, the ordinate of the point of inflexion sought.

EXAM. 2. To find the point of inflexion in a curve defined by the equation $ay = a\sqrt{ax} + x^2$.

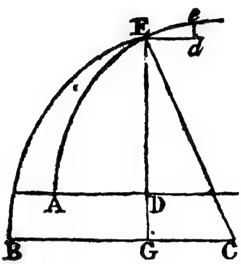
EXAM. 3. To find the point of inflexion in a curve defined by the equation $ay^2 = a^2x + x^3$.



EXAM. 4. To find the point of inflexion in the Conchoid of Nicomedes, which is generated or constructed in this manner: From a fixed point P, which is called the pole of the conchoid, draw any number of right lines PA, PB, PC, PE, &c. cutting the given line FD in the points F, G, H, I, &c.; then make the distances FA, GB, HC, IE, &c. equal to each other, and equal to a given line; then the curve line ABCE, &c. will be the conchoid; a curve so called by its inventor Nicomedes.

TO FIND THE RADIUS OF CURVATURE OF CURVES.

81. THE Curvature of a Circle is constant, or the same in every point of it, and its radius is the radius of curvature. But the case is different in other curves, every one of which has its curvature continually varying, either increasing or decreasing, and every point having a degree of curvature peculiar to itself; and the radius of a circle which has the same curvature with the curve at any given point, is the radius of curvature at that point; which radius it is the business of this chapter to find.



82. Let AEe be any curve, concave towards its axis AD; draw an ordinate DE to the point E, where the curvature is to be found; and suppose EC perpendicular to the curve, and equal to the radius of curvature sought, or equal to the radius of a circle having the same curvature there, and with that radius describe the said equally curved circle BEe; lastly, draw Ed parallel to AD, and dc parallel and indefinitely near to DE: thereby making Ed the fluxion or increment of the absciss AD, also de the fluxion of the ordinate DE, and Ee that of the curve AE. Then put $x =$ AD, $y =$ DE, $z =$ AE, and $r =$ CE the radius of curvature; then E$d = \dot{x}$, $de = \dot{y}$, and E$e = \dot{z}$.

Now, by sim. triangles, the three lines Ed, de, Ee,
which vary as $\dot{x}, \dot{y}, \dot{z}$,
are respectively as the three - - GE, GC, CE;
therefore - - - - - GC . $\dot{x}$ = GE . $\dot{y}$;

and the flux. of this eq. is GC . $\ddot{x}$ + $\dot{\text{GC}}$. $\dot{x}$ = GE . $\ddot{y}$ + $\dot{\text{GE}}$. $\dot{y}$,

or because $\dot{\text{GC}} = -\dot{\text{BG}}$, it is GC . $\ddot{x}$ − $\dot{\text{BG}}$. $\dot{x}$ = GE . $\ddot{y}$ + $\dot{\text{GE}}$. $\dot{y}$.

But since the two curves AE and BE have the same curvature at the point E, their abscisses and ordinates have the same fluxions at that point, that is, Ed or $\dot{x}$ is the fluxion both of AD and BG, and de or $\dot{y}$ $\propto$ the fluxion both of DE and GE. In the equation above therefore substitute $\dot{x}$ for $\dot{\text{BG}}$, and $\dot{y}$ for $\dot{\text{GE}}$, and it becomes

$$\text{GC}\ddot{x} - \dot{x}\dot{x} = \text{GF}\ddot{y} + \dot{y}\dot{y},$$
$$\text{or GC}\ddot{x} - \text{GF}\ddot{y} = \dot{x}^2 + \dot{y}^2 = \dot{z}^2.$$

Now multiply the three terms of this equation respectively by these three quantities, $\frac{\dot{y}}{GC}$, $\frac{\dot{x}}{GE}$, $\frac{\dot{z}}{CE}$, which are all equal,

and it becomes $\dot{y}\ddot{x} - \dot{x}\ddot{y} = \frac{\dot{z}^3}{CE}$, or $\frac{\dot{z}^3}{r}$;

and hence is found $r = \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$, for the general value of the radius of curvature, for all curves whatever, in terms of the fluxions of the absciss and ordinate.

83. Further, as in any case either x or y may be supposed to flow equably, that is, either $\dot{x}$ or $\dot{y}$ constant quantities, or $\ddot{x}$ or $\ddot{y}$ equal to nothing, it follows that, by this supposition, either of the terms of the denominator, of the value of r, may be made to vanish. Thus, whon $\dot{x}$ is supposed constant, $\ddot{x}$ being then $= 0$, the value of r is barely $\frac{\dot{z}^3}{-\dot{x}\ddot{y}}$; or r is $= \frac{\dot{z}^3}{\dot{y}\ddot{x}}$ when $\dot{y}$ is constant.

EXAMPLES.

Exam. 1. To find the radius of curvature to any point of a parabola, whose equation is $ax = y^2$, its vertex being A, and axis AD.

Here, the equation to the curve being $ax = y^2$, the fluxion of it is $a\dot{x} = 2y\dot{y}$; and the fluxion of this again is $a\ddot{x} = 2\dot{y}^2$, supposing $\dot{y}$ constant; hence then r or

$$\frac{\dot{z}^3}{\dot{y}\ddot{x}} \text{ or } \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{\dot{y}\ddot{x}} \text{ is } = \frac{(a^2 + 4y^2)^{\frac{3}{2}}}{2a^2} \text{ or } \frac{(a + 4x)^{\frac{3}{2}}}{2\sqrt{a}},$$

for the general value of the radius of curvature at any point E, the ordinate to which cuts off the absciss AD $= x$.

Hence, when the absciss x is nothing, the last expression becomes barely $\frac{1}{2}a = r$, for the radius of curvature at the vertex of the parabola; that is, the diameter of the circle of curvature at the vertex of a parabola, is equal to a, the parameter of the axis. See, also, pa. 139.

Exam. 2. To find the radius of curvature of an ellipse, whose equation is $a^2y^2 = c^2(ax - x^2)$.

$$\text{Ans. } r = \frac{[a^2c^2 + 4(a^2 - c^2) \times (ax - x^2)]^{\frac{3}{2}}}{2a^4c}.$$

Exam. 3. To find the radius of curvature of an hyperbola, whose equation is $a^2y^2 = c^2(ax + x^2)$,

$$\text{Ans. } r = \frac{[a^2c^2 + 4(a^2 + c^2) \times (ax + x^2)]^{\frac{3}{2}}}{2a^4c}.$$

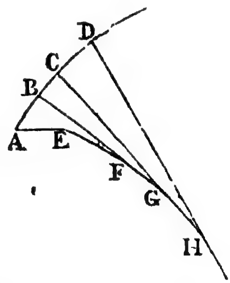
Exam. 4. To find the radius of curvature of the cycloid.

Ans. $r = 2\sqrt{(aa - ax)}$, where x is the absciss, and a the diameter of the generating circle.

OF INVOLUTE AND EVOLUTE CURVES.

84. An Evolute is any curve supposed to be evolved or opened, which having a thread wrapped close about it, fastened at one end, and beginning to evolve or unwind the thread from the other end, keeping always tight stretched the part which is evolved or wound off: then this end of the thread will describe another curve, called the Involute. Or, the same involute is described in the contrary way, by wrapping the thread about the curve of the evolute, keeping it at the same time always stretched.

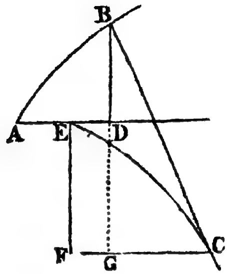
85. Thus, if EFGH be any curve, and AE be either a part of the curve, or a right line: then if a thread be fixed to the curve at H, and be wound or plied close to the curve, &c. from H to A, keeping the thread always stretched tight; the other end of the thread will describe a certain curve ABCD, called an Involute; the first curve EFGH being its evolute. Or, if the thread, fixed at H, be unwound from the curve, beginning at A, and keeping it always tight, it will describe the same involute ABCD.



86. If AE, BF, CG, DH, &c. be any positions of the thread, in evolving or unwinding; it follows, that these parts of the thread are always the radii of curvature, at the corresponding points, A, B, C, D; and also equal to the corresponding lengths AE, AEF, AEFG, AEFGH, of the evolute; that is,

AE = AE is the radius of curvature to the point A,
BF = AEF is the radius of curvature to the point B,
CG = AEG is the radius of curvature to the point C,
DH = AEH is the radius of curvature to the point D.

87. It also follows, from the premises, that any radius of curvature, BF, is perpendicular to the involute at the point B, and is a tangent to the evolute curve at the point F. Also, that the evolute is the locus of the centre of curvature of the involute curve.



88. Hence, and from art. 82, it will be easy to find one of these curves, when the other is given. To this purpose, put

x = AD, the absciss of the involute,
y = DB, an ordinate to the same,
z = AB, the involute curve,
r = BC, the radius of curvature,
v = EF, the absciss of the evolute EC,
u = FC, the ordinate of the same, and
a = AE, a certain given line.

Then by the nature of the radius of curvature, it is

$r = \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$ = BC = AE + EC; also, by sim. triangles,

$$\dot{z} : \dot{x} :: r : \text{GB} = \frac{r\dot{x}}{\dot{z}} = \frac{\dot{x}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} = \frac{\dot{z}^2}{-\ddot{y}};$$

$$\dot{z} : \dot{y} :: r : \text{GC} = \frac{r\dot{y}}{\dot{z}} = \frac{\dot{y}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} = \frac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}}.$$

Hence $\text{EF} = \text{GB} - \text{DB} = \frac{\dot{z}^2}{-\ddot{y}} - y = v$;

and $\text{FC} = \text{AD} - \text{AE} + \text{GC} = x - a + \frac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}} = u$;

which are the values of the absciss and ordinate of the evolute curve EC; from which therefore these may be found, when the involute is given.

On the contrary, if v and u, or the evolute, be given: then, putting the given curve EC = s, since CB = AE + EC, or $r = a + s$, this gives r the radius of curvature. Also, by similar triangles, there arise these proportions, viz.

$$\dot{s} : \dot{v} :: r : \frac{r\dot{v}}{\dot{s}} = \frac{\overline{a+s}\cdot\dot{v}}{\dot{s}} = \text{GB},$$

and $\dot{s} : \dot{u} :: r : \frac{r\dot{u}}{\dot{s}} = \frac{\overline{a+s}\cdot\dot{u}}{\dot{s}} = \text{GC}$;

theref. $\text{AD} = \text{AE} + \text{FC} - \text{GC} = a + u - \frac{\overline{a+s}\cdot\dot{u}}{\dot{s}} = x$,

which are the absciss and ordinate of the involute curve, and which may therefore be found, when the evolute is given. Where it may be noted, that $\dot{s}^2 = \dot{v}^2 + \dot{u}^2$, and $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$. Also, either of the quantities x, y, may be supposed to flow equably, in which case the respective second fluxion, $\ddot{x}$ or $\ddot{y}$, will be nothing, and the corresponding term in the denominator $\dot{y}\ddot{x} - \dot{x}\ddot{y}$ will vanish, leaving only the other term in it; which will have the effect of rendering the whole operation simpler.

EXAMPLES.

Exam. 1. To determine the nature of the curve by whose evolution the common parabola AB is described.

Here the equation of the given involute AB, is $cx = y^2$ where c is the parameter of the axis AD. Hence then $y = \sqrt{cx}$, and $\dot{y} = \frac{1}{2}\dot{x}\sqrt{\frac{c}{x}}$, also $\ddot{y} = \frac{-\dot{x}^2}{4x}\sqrt{\frac{c}{x}}$ by making $\dot{x}$ constant. Consequently the general values of v and u, or of the absciss and ordinate, EF and FC, above given, become, in that case,

$$\text{EF} = v = \frac{\dot{z}^2}{-\ddot{y}} - y = \frac{\dot{x}^2 + \dot{y}^2}{-\ddot{y}} - y = 4x\sqrt{\frac{x}{c}}; \text{ and}$$

$$\text{FC} = u = x - a + \frac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}} = 3x + \tfrac{1}{2}c - a.$$

But the value of the quantity a or AE, by exam. 1 to art. 83, was found to be $\frac{1}{2}c$; consequently the last quantity, FC or u, is barely $= 3x$.

Hence then, comparing the values of v and u, there is found $3v\sqrt{c} = 4u\sqrt{x}$, or $27cv^2 = 16u^3$; which is the equation between the absciss and ordinate of the evolute curve EC, showing it to be the semicubical parabola.

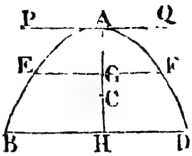
Exam. 2. To determine the evolute of the common cycloid.

Ans. another cycloid, equal to the former.

TO FIND THE CENTRE OF GRAVITY.

89. By referring to art. 108, &c. in the *Statics*, it is seen what are the principles and nature of the Centre of Gravity

in any figure, and how it is generally expressed. It there appears, that if PAQ be a line, or plane, drawn through any point, as suppose the vertex of any body, or figure, ABD, and if s denote any section EF of the figure; d = AG, its distance below PQ, and b = the whole body or figure ABD: then the distance AC, of the centre of



gravity below PQ, is universally denoted by $\frac{\text{sum of all the } ds}{b}$; whether ABD be a line, or a plane surface, or a curve superficies, or a solid.

But the sum of all the ds, is the same as the fluent of $d\dot{b}$, and b is the same as the fluent of $\dot{b}$; therefore the general expression for the distance of the centre of gravity, is AC = $\frac{\text{fluent of } x\dot{b}}{\text{fluent of } \dot{b}} = \frac{\text{fluent } x\dot{b}}{b}$; putting $x = d$ the variable distance AG. Which will divide into the following four cases.

90. Case 1. When AE is some line, as a curve suppose. In this case $\dot{b}$ is $= \dot{z}$ or $\sqrt{\dot{x}^2 + \dot{y}^2}$, the fluxion of the curve; and $b = z$: theref. AC $= \frac{\text{fluent of } x\dot{z}}{z} = \frac{\text{fluent of } x\sqrt{\dot{x}^2 + \dot{y}^2}}{z}$ $= \frac{fx\dot{z}}{z}$ is the distance of the centre of gravity in a curve.

91. Case 2. When the figure ABD is a plane; then $\dot{b} = y\dot{x}$; therefore the general expression becomes AC = $\frac{fyx\dot{x}}{fy\dot{x}}$ for the distance of the centre of gravity in a plane.

92. Case 3. When the figure is the superficies of a body generated by the rotation of a line AEB, about the axis AH. Then putting $\pi = 3{\cdot}14159$, &c. $2\pi y$ will denote the circumference of the generating circle, and $2\pi y\dot{z}$ the fluxion of the surface; therefore AC $= \frac{\text{fluent of } 2\pi yx\dot{z}}{\text{fluent of } 2\pi y\dot{z}} = \frac{fyx\dot{z}}{fy\dot{z}}$ will be the distance of the centre of gravity for a surface generated by the rotation of a curve line z.

93. Case 4. When the figure is a solid generated by the rotation of a plane ABH, about the axis AH.

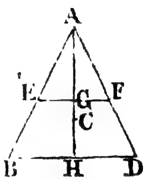
Then is $\pi y^2 =$ the area of the circle whose radius is y, and $\pi y^2\dot{x} = \dot{b}$, the fluxion of the solid; therefore - - -

$$\text{AC} = \frac{\text{fluent of } x\dot{b}}{\text{fluent of } \dot{b}} = \frac{\text{fluent of } \pi y^2 x\dot{x}}{\text{fluent of } \pi y^2\dot{x}} = \frac{fy^2x\dot{x}}{fy^2\dot{x}}$$ is the distance of the centre of gravity below the vertex in a solid.

EXAMPLES.

EXAM. 1. Let the figure proposed be the isosceles triangle ABD.

It is evident that the centre of gravity C, will be somewhere in the perpendicular AH. Now, if a denote AH, $c =$ BD, $x =$ AG, and $y =$ EF any line parallel to the base BD: then as $a : c :: x : y = \frac{cx}{a}$; therefore, by the 2d Case, $\text{AC} = \frac{\text{fluent } yx\dot{x}}{\text{fluent } y\dot{x}} = \frac{\text{fluent } x^2\dot{x}}{\text{fluent } x\dot{x}} = \frac{\frac{1}{3}x^3}{\frac{1}{2}x^2}$ $= \frac{2}{3}x = \frac{2}{3}\text{AH}$, when x becomes $=$ AH: consequently CH $= \frac{1}{3}$AH.



In like manner, the centre of gravity of any other plane triangle, will be found to be at $\frac{2}{3}$ of the altitude of the triangle; the same as it was found in art. 111, *Statics*.

EXAM. 2. In a parabola; the distance from the vertex is $\frac{3}{5}x$, or $\frac{3}{5}$ of the axis.

EXAM. 3. In a circular arc; the distance from the centre of the circle, is $\frac{cr}{a}$; where a denotes the arc, c its chord, and r the radius.

EXAM. 4. In a circular sector; the distance from the centre of the circle, is $\frac{2cr}{3a}$: where a, c, r, are the same as in exam. 3.

EXAM. 5. In a circular segment; the distance from the centre of the circle is $\frac{c^3}{12a}$; where c is the chord, and a the area, of the segment.

EXAM. 6. In a cone, or any other pyramid; the distance from the vertex is $\frac{3}{4}x$, or $\frac{3}{4}$ of the altitude.

EXAM. 7. In the semisphere, or semispheroid; the distance

from the centre is $\frac{3}{8}r$; or $\frac{3}{8}$ of the radius: and the distance from the vertex $\frac{5}{8}$ of the radius.

EXAM. 8. In the parabolic conoid; the distance from the base is $\frac{1}{3}x$, or $\frac{1}{3}$ of the axis. And the distance from the vertex $\frac{2}{3}$ of the axis.

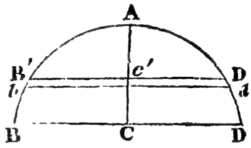
EXAM. 9. In the segment of a sphere, or of a spheroid; the distance from the base is $\frac{2a - x}{6a - 4x}x$; where x is the height of the segment, and a the whole axis, or diameter of the sphere.

EXAM. 10. In the hyperbolic conoid; the distance from the base is $\frac{2a + x}{6a + 4x}x$; where x is the height of the conoid, and a the whole axis or diameter.

94. Among the preceding examples, those which relate to circles and spheres, furnish pleasing applications of the fluxional formulæ for the trigonometrical quantities. Thus

1. To find the centre of gravity of a circular arc.

Let AB$' = z$, B$'b = \dot z$,
rad. $= 1$.



Then $\frac{\int x\dot z}{z} = \frac{\int 2\cos. z\dot z}{2z} = \frac{2\sin. z}{2z}$.

Hence, BAD : BD : : rad. : dist. C. G. from A.

2. To find the centre of gravity of a circular segment.

$$\text{Here } \frac{\int xy\dot x}{\int y\dot x} = \frac{\int 2\sin. z\cos. z\,\phi\cos. z}{\int 2\sin. z\,\phi\cos. z} = \frac{\int 2\sin.^2 z\cos. z\dot z}{\int 2\sin.^2 z\dot z}$$

$$= \frac{\int 2\sin.^2 z\,\phi\sin. z}{\int(1 - \cos. 2z)\dot z} = \frac{\frac{2}{3}\sin.^3 z}{z - \frac{1}{2}\sin. 2z}.$$

3. To find the distance of C. G. of spheric surface from its centre.

$$\frac{\int 2\pi\sin. z\cos. z\dot z}{\int 2\pi\sin. z\dot z} = \frac{\pi\sin.^2 z}{2\pi(1 - \cos. z)} = \tfrac{1}{2}(1 + \cos. z):$$

that is, at the middle point of the versed sine.

4. For the C. G. of a spherical segment.

$$\frac{\int y^2 x\dot x}{\int y^2\dot x} = \frac{\int\sin.^2 z\cos. z\,(-\phi\cos. z)}{\int\sin.^2 z\,(-\phi\cos. z)} = \frac{\int\sin.^2 z\cos. z\,\phi\cos. z}{\int\sin.^2 z\,\phi\cos. z}$$

$$= \frac{\int\sin.^2 z\cos. z\sin. z\dot z}{\int\sin.^2 z\,(-\phi\cos. z)} = \frac{\int\sin.^3 z\cos. z\dot z}{\int\sin.^2 z\,(-\phi\cos. z)}$$

$$-\frac{f\sin.^3z\ \varphi\sin.z}{f\sin.^2z\,(-\varphi\cos.z)} = f(1-\cos.^2z)(-\varphi\cos.z)$$

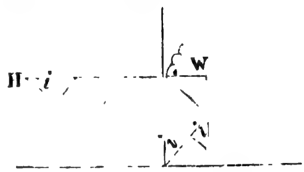
$$= \frac{\frac{1}{4}\sin.^4z}{f(\cos.^2z-1)\ \varphi\cos.z} = \frac{\frac{1}{4}\sin.^4z}{\frac{1}{3}(\cos.^3z-3\cos.z)},$$

which corrected becomes $= \dfrac{\frac{1}{4}\sin.^4z}{\frac{1}{3}(\cos.^3z-3\cos.z+2)}$.

When the segment becomes a hemisphere, this becomes $=\frac{3}{8}$ of radius, as it ought to be.

95. *Pressure of Earth against Walls.*

Lemma. A weight, W, being placed on a plane, inclined to the vertical in an angle i, to find a horizontal force, H, sufficient to sustain it, so that it shall not run down the plane, taking friction into the account.



Each of the forces, W, H, being resolved into two, the one parallel, the other perpendicular, to the plane; there will result,

parallel to the plane, a force $= \text{W}\cos.i - \text{H}\sin.i$,
perp. to the plane, a force $= \text{W}\sin.i + \text{H}\cos.i$.

In order to an equilibrium, the first of these forces ought to be precisely equal to the friction down the plane.

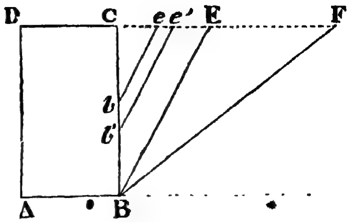
That is, $\text{W}\cos.i - \text{H}\sin.i = f\text{W}\sin.i + f\text{H}\cos.i$,
whence $f\text{H}\cos.i + \text{H}\sin.i = -f\text{W}\sin.i + \text{W}\cos.i$,

and $\text{H} = \text{W}\dfrac{\cos.i - f\sin.i}{\sin.i + f\cos.i} = \text{W}\dfrac{1-f\tan.i}{\tan.i+f}$.

Corol. Hence, if instead of a horizontal force, the weight W were sustained by a wall, or by any obstacle whatever, the horizontal effort exerted by the weight against the obstacle would be $\text{W}.\dfrac{1-f\tan.i}{\tan.i+f}$.

96. Prop. To determine the horizontal stress of the terrace whose vertical section is BCEF, against the wall whose section is ABCD, and the momentum of the pressure to overturn the wall about the angle A.

Considering, first, the stress of a triangle CBE, whose sloping side BE makes the angle i with the vertical: let bc, $b'c'$, be each parallel to BE, limiting the elementary trapezoid



$bb'e'e$. Let $BC = a$, $cb = x$, $bb' = \dot{x}$; then area of $bb'e'e = x\dot{x}$ tan. i; and if s be the specific gravity of the earth, the weight of the portion $bb'e'e$ will be $= sx\dot{x}$ tan. i. Therefore the horizontal effort, against the line bb', will be

$$= sx\dot{x} \text{ tan. } i \cdot \frac{1 - f \text{ tan. } i}{\text{tan. } i + f} = sx\dot{x} \frac{1 - f \text{ tan. } i}{1 + f \text{ cot. } i}$$

$$= sx\dot{x}\text{M} ; \text{ putting } \frac{1 - f \text{ tan. } i}{1 + f \text{ tan. } i} = \text{M}.$$

The fluent of $sx\dot{x}\text{M}$, when $x = a$, gives $\frac{1}{2}a^2 s\text{M}$, for the whole horizontal thrust of the triangle CBE.

Referring the momentum of the thrust of the elementary portion $bb'e'e$, to the length of lever $b\text{B} = a - x$, we have for that momentum $\text{M}s\,(a - x)\,x\dot{x}$. The fluent of this when $x = a$, is $= \frac{1}{6}a^3 s\text{M}$.

97. It remains to determine the angle i.

Now, it is evident that $\frac{1 - f \text{ tan. } i}{1 + f \text{ cot. } i} = \text{M}$, vanishes, and consequently both the horizontal thrust and its momentum, vanish, whether tan. $i = 0$, or $= \frac{1}{f}$. Between these two values, therefore, there is one which gives both the greatest thrust and the greatest momentum. This value is found by making

$$\varphi\text{M} = 0, \text{ that is, } \varphi \frac{1 - f \text{ tan. } i}{1 + f \text{ cot. } i} = 0. \quad \text{Put tan. } i = z,$$

$$\text{then } -f\dot{z}\left(1 + \frac{f}{z}\right) + \frac{f\dot{z}}{z^2}(1 - fz) = 0;$$

$$\text{or } \dot{z} + \frac{f\dot{z}}{z} = \frac{\dot{z} - fz\dot{z}}{z^2},$$

$$1 + \frac{f}{z} = \frac{1 - fz}{z^2} \quad . \quad z^2 + fz = 1 - fz,$$

$$z^2 + 2fz = 1 \;.\;.\; z^2 + 2fz + f^2 = 1 + f^2,$$

$$z + f = \sqrt{1 + f^2} \quad . \; . \; . \; . \quad z = -f + \sqrt{1 + f^2},$$

that is,

$$\text{tan. } i = -f + \sqrt{1 + f^2}.$$

Substituting this value of tan. i for it in the above expression for M, we have for the horizontal thrust

$$\tfrac{1}{2}a^2s\{-f+\sqrt{(1+f^2)}\}^2 = \tfrac{1}{2}a^2s\tan.^2 i,$$

while the momentum of the stress is found to be

$$\tfrac{1}{6}a^3s\{-f+\sqrt{(1+f^2)}\}^2 = \tfrac{1}{6}a^3s\tan.^2 i,$$

which was to be found.

98. The angle which has for its tangent $\frac{1}{f}$ is the angle of the slope, which the earth would, of itself, naturally take, if it were not sustained by any wall.

For a body has a tendency to descend along a plane (inclination to vertical $= i$) with a force $= g$ cos. i, and it presses the plane with a force $= g$ sin. i. Wherefore the friction $= fg$ sin. i; and since it counterbalances the force with which the body endeavours to descend, we have

$$fg\sin.\, i = g\cos.\, i \therefore \frac{\sin.\, i}{\cos.\, i} = \tan.\, i = \frac{1}{f};$$

$$\text{also } f = \cot.\, i.$$

Farther, the angle whose tangent is $-f+\sqrt{1+f^2}$ is half the angle whose tangent is $\frac{1}{f}$.

For $\tan.\, i = \dfrac{2\tan.\frac{1}{2}i}{1-\tan.^2\frac{1}{2}i}$. (Equa. 17, pa. 18.)

$$\text{Or, } \frac{2[-f+\sqrt{1+f^2}]}{1-[-f+\sqrt{1+f^2}]^2} = \frac{-2f+2\sqrt{1+f^2}}{-2f^2+2f\sqrt{1+f^2}} = \frac{1}{f}.$$

Let, therefore, BF be the slope which loose earth would, of itself, naturally assume: then, the line BE which determines the triangle of earth that exerts the greatest horizontal stress against the vertical wall bisects the angle CBF.

99. SCHOLIUM.

Sandy and loose earth, takes a natural declivity of 60° from the vertical; stronger earth will take a declivity of 53°. Therefore, for a terrace of loose earth we have $i = 30°$; for another of strong and close earth $i = 26\frac{1}{2}°$.

Hence, for the former kind, where tan. $30° = \frac{1}{3}\sqrt{3}$, the value of the stress is $\frac{1}{6}a^2s$, and that of the momentum of the stress $\frac{1}{18}a^3s$.

For the latter kind, where tan. $26\frac{1}{2}° = \frac{1}{2}$ nearly, the stress $= \frac{1}{8}a^2s$, its momentum $= \frac{1}{24}a^3s$.

100. The horizontal stress and momentum being thus known, it is easy to proportion to them the resistance of the wall ABCD.

Let b = AB, while BC = a, and let s be the spec. grav. of the wall. For brick $s = 2000$, for strong earth, $s = 1428$. Then the momentum of the resistance referred to the point AB, being $\frac{1}{2}ab^2\text{s}$; we shall have

$$\tfrac{1}{2}ab^2s = \tfrac{1}{24}a^3s \text{ (for strong earth)}$$

$$\therefore b = a\sqrt{\frac{s}{12s}} = \cdot 28034 \times a\sqrt{\frac{s}{\text{s}}}.$$

Thus, if $a = 39{\cdot}37$ feet, s and s as above we shall find $b = 9{\cdot}326$ feet.

EXAM. 2. Supposing the earth of the same kind as in the above example, s to s, as 4 to 5, and the height of the wall and bank each 12 feet; required the thickness of the wall, being rectangular. Ans. 2·986 feet.

Note. The preceding investigation proceeds upon the principles assumed by *Coulomb* and *Prony*. They who wish to go thoroughly into this subject, and have not opportunity to make experiments, may advantageously consult *Traité Expérimental, Analytique et Pratique de la Poussée des Terres, &c. par M. Mayniel.*

ON THE FLEXIBILITY, STRENGTH, AND RUPTURE OF TIMBER, &c.

A piece of solid matter may be exposed to, at least, four distinct kinds of strains: viz.

1st. It may be pulled, or torn, asunder, as in the case of ropes, stretchers, king-posts, tie-beams, &c.

2dly. It may be crushed, as in the case of pillars, posts, and truss-beams.

3dly. It may be broken across, as in the case of a joist or rafter.

4thly. It may be wrenched, or twisted, as in the case of the axle of a wheel, the nail of a press, &c.

The complete investigation of these particulars, only in their principal varieties, would require a volume. The student who wishes to go into the inquiry with scientific precision, may consult M. Girard's Treatise on the Resistance of Solids, an interesting essay on the Flexibility of Wood, by M. Dupin, in *Journal de l'Ecole Polytechnique*, tome 10, Tredgold's Principles of Carpentry, and Mr. Barlow's valuable *Essay on the Strength and Stress of Timber*. Having attended many of the experiments recorded in the latter-mentioned work, I can with confidence recommend its principal results as accurate and useful; and shall, therefore, refer to the work itself for the experiments and investigations from which the following formulæ and rules are deduced.

Let l denote the length, a the breadth, d the depth of a rectangular beam, all in inches, W the weight with which it is loaded in the middle, (being supported at both ends), δ the deflection occasioned by that weight, and E the measure of the elasticity: then it is found that $\frac{\mathrm{W}l^3}{ad^3\delta} = \mathrm{E}$ is a constant quantity, for the same timber; or, which amounts to the same, that $\frac{\mathrm{W}l^3}{\mathrm{E}ad^3} = \delta$.

This formula is equally applicable to beams fixed at one end, and loaded at the other, and those which are supported at both ends and loaded in the middle; only the value of E in the one case will be to that in the other, as 32 to 1.

For the ultimate deflection of beams before their rupture, the theorem is $\frac{l^2}{d\Delta} = \mathrm{U}$, where Δ is the last deflection.

If the resistance of a rod an inch square be S, then $ad^2\mathrm{S}$ will be the resistance of a beam the same length, whose breadth is a and depth d: also, if the angle of deflection be Δ, and the breaking weight be W; then

1. *When the beam is fixed at one end, and loaded at the other.*

$$l\mathrm{W} \cos. \Delta = ad^2\mathrm{S}, \text{ or } \frac{l\mathrm{W} \cos. \Delta}{ad^2} = \mathrm{S}, \text{ a constant quantity.}$$

2. When the beam is supported at each end, and loaded in the middle.

$$\tfrac{1}{4}l\mathrm{W}sec^2\Delta = ad^2\mathrm{S}, \text{ or } \frac{l\mathrm{W}sec^2\Delta}{4ad^2} = \mathrm{S}, \textit{ constant.}$$

3. *When the beam is fixed at each end, and loaded in the middle.*

$$\tfrac{1}{6} l w s e c^2 \triangle = a d^2 s, \text{ or } \frac{l w s e c^2 \triangle}{6 a d^2} = s, \textit{ constant.}$$

4. *When the beam in either of the two last cases is loaded at any other point than the centre.*

We shall have, in the former case, by denoting the two unequal lengths by m and n,

$$\frac{m n w}{l} s e c^2 \triangle = a d^2 s, \text{ or } \frac{m n w s e c^2 \triangle}{l a d^2} = s:$$

and in the second,

$$\frac{2 m n w}{3 l} s e c^2 \triangle = a d^2 s, \text{ or } \frac{2 m n w s e c^2 \triangle}{3 l a d^2} = s,$$

still the same constant quantity.

And the first formula will also apply to a beam fixed at any given angle of inclination; observing only, that the angle $\triangle$, in this case, will represent the angle of the beam's inclination, increased or diminished by the angle of its deflection, according as its first position is ascending or descending; or rather, it will denote the angle of the beam's inclination at the moment of fracture.

In all these cases, when it is only intended to apply the results to the common application of timber to architectural and other purposes, the angle of deflection may be omitted, and the equations then become simply,

1. $\dfrac{l w}{a d^2} = s,$ 2. $\dfrac{l w}{4 a d^2} = s,$

3. $\dfrac{l w}{6 a d^2} = s,$ 4. $\dfrac{m n w}{l a d^2} = s,$

5. $\dfrac{2 m n w}{3 l a d^2} = s.$

The absolute value of direct cohesion on a square inch is $c = \dfrac{s' d^2}{(d - D)^2}$; where D is the depth of the neutral axis, or of the line which separates the compressed from the stretched portion of the wood.

The subjoined table of data for different kinds of wood, results from the union of these formulæ with experiments.

Name of the kind of Wood.	Spec. Grav.	Value of U.	Value of E.	Value of S.	Value of S′.	Value of C.
Teak	745	818	9657802	2462	2488	15555
Poon	579	596	6759200	2221	2266	14787
Eng. Oak	969	598	3494730	1181	1205	9836
Do. Spec. 2. ..	934	435	5806200	1672	1736	10853
Canadian Oak	872	588	8595864	1766	1803	11428
Dantzic Oak ..	756	724	4765750	1457	1477	7386
Adriatic Oak ..	993	610	3885700	1583	1409	8808
Ash..........	760	395	6580750	2026	2124	17337
Beech	696	615	5417266	1556	1586	9912
Elm	553	509	2799347	1013	1042	5767
Pitch Pine	660	588	4900466	1632	1666	10415
Red Pine	657	605	7359700	1341	1368	10000
New Eng. Fir..	553	757	5967400	1102	1116	9917
Riga Fir......	753	588	5314570	1108	1131	10707
Do. Spec. 2. ..	738	..	3962800	1051	1081	....
Mar Forest Fir	696	588	2581400	1144	1168	9539
Do. Spec. 2. ..	693	403	3478328	1262	1310	10691
Larch	531	411	2465433	653	890	....
Do. Spec. 2. ..	522	518	3591133	832	850	....
Do. Spec. 3. ..	556	518	4210830	1127	1119	7655
Do. Spec. 4. ..	560	518	4210830	1149	1172	7352
Norway Spar..	577	618	5832000	1471	1492	12180

Other tables and observations on the cohesive strength of metals, &c. are given at the end of this volume.

Solution of Practical Problems, from the preceding Data.

PROB. I. *To find the Strength of Direct Cohesion of a Piece of Timber of any given Dimensions.*

Rule.—Multiply the area of the transverse section, in inches, by the value of C, in the preceding table of data, and the product will be the strength required.

Note. If the specific gravity be not the same as the mean tabular specific gravity; say, as the latter is to the former, so is the above product to the correct result.

EXAM. 1. What weight will it require to tear asunder a piece of teak 3 inches square, the specific gravity being 745?

Ans. 139·95lbs.

EXAM. 2. What weight will break vertically a cylinder of ash, 2 inches in diameter, and specific gravity 700?

Ans. 50166lbs.

PROB. II. *To compute the Deflection of Beams fixed at one End and loaded at the other with any given Weight.*

Rule 1. Multiply the tabular value of E by the breadth and cube of the depth of the given beam, both in inches.

2. Multiply also the cube of the length in inches by the given weight, and that product again by 32.

3. Divide the latter product by the former, for the deflection sought.

EXAM. 1. An ash batten, 3 inches square, is fixed in a wall, and projects from it 4 feet. If a weight of 200lbs. be hung on its extremity, how much will it be deflected?

Ans. $1\frac{1}{3}$ inches.

EXAM. 2. What would the same beam be deflected if a prop or shore, proceeding from the wall, met it at half its length?

Here, without repeating the operation, as we know that the deflections are as the cubes of the lengths; and as by means of the shore the length is reduced to one half the former, viz. to 2 feet, we have

$$4^3 : 2^3 :: 1\tfrac{1}{3} \text{ inches} : \text{(former deflec.)}$$

$$\frac{1\frac{1}{3} \times 2^3}{4^3} = \frac{1\frac{1}{3}}{8} = \frac{4}{24} = \tfrac{1}{6} \text{ of an inch, } \textit{answer}.$$

EXAM. 3. A batten of New England fir, 6 feet long and 4 inches deep, by $2\frac{1}{2}$ inches in breadth, is fixed at one end, and loaded, uniformly throughout its length, with 200lbs., how much will its extremity be deflected?

Note. The same rule will apply, when the weight is distributed throughout the length, by multiplying the second product by 12 instead of 32.

PROB. III. *To compute the Deflection of Beams, supported at each End, and loaded in the Middle with any given Weight.*

Rule 1. Multiply the tabular value of E by the breadth and cube of the depth, both in inches.

2. Multiply also the cube of the length, in inches, by the given weight in lbs.; then divide the latter product by the former for the deflection sought.

EXAM. 1. A square beam of English oak, whose side is 6 inches, is supported on two walls, 20 feet distant, and is to be loaded at its middle point with 1000lbs., what will it be deflected?

Ans. 1·8 inch.

EXAM. 2. A beam of red pine, 8 inches in breadth, and

1 foot deep, is supported on two walls, distant 33 feet 4 inches: how much will it be deflected with 2000lbs. suspended at its centre? Ans. $1\frac{1}{4}$ inches.

Note. If the beam be *fixed* at each end, the deflection will, with equal weights, be two-thirds of that found by the above rule.

PROB. IV. *To compute the Deflection of Beams supported at each end, and loaded uniformly throughout their Length with a given Weight.*

Rule. Compute the deflection the same as in the last problem. Multiply that result by 5, and divide the product by 8, and the quotient will be the answer.

EXAM. 1. A uniform bar of Adriatic oak, 2 inches square, is rested upon two props, distant 24 feet, how much will it be deflected by its own weight, its specific gravity being 960, or 60lbs. to the cubic foot? Ans. $9\frac{1}{2}$ inches.

EXAM. 2. A beam of Riga fir, 12 inches square, is to support the brick work over a gateway, 12 feet wide; the computed weight of the brick work is 30000lbs., what deflection may be expected? Ans. ·58 inch.

PROB. V. *To compute the ultimate Deflection of Beams, or Rods, before their Rupture.*

Note. The beams are supposed to be supported at each end.

Rule. Multiply the tabular value of U, in the preceding table of data, by the depth of the beam in inches, and divide the square of the length, also in inches, by that product, for the ultimate deflection sought.

EXAM. A square inch rod of ash, 6 feet long, is broken by a weight applied to its centre: how much will it be deflected before it breaks? Ans. 13·1 inches.

PROB. VI. *To find the ultimate transverse Strength of any rectangular Beam of Timber, fixed at one End and loaded at the other.*

Rule I. Multiply the value of S, in the preceding table of data, by the breadth and square of the depth, both in inches, and divide that product by the length, also in inches, and the quotient will be the weight in lbs. This is approximative.

Rule II. 1. Take the ultimate deflection 8 times that of the last problem, and divide the deflection by the length,

which will give the sine of the angle of deflection; whence, by a table, find the secant.

2. Multiply this secant by the breadth and square of the depth in inches, and the product again by the value of s' in the table of data.

3. Divide this last product by the length in inches, and the quotient will be the answer, in lbs.

EXAM. 1. What weight will it require to break a piece of Mar forest fir, fixed by one end in a wall, and loaded at the other; the breadth being 2 inches, depth 3 inches, and length 4 feet? Ans. 518lbs.

EXAM. 2. A square oaken balk, 12 inches square, projects 8 feet 4 inches from a solid wall, in which it is fixed; what weight will be sufficient to break it? Ans. 50345lbs.

EXAM. 3. A piece of ash, 2 inches square, projects 6 feet from a wall in which it is fixed; what weight, uniformly distributed through its length, will be required to break it?

PROB. VII. *To compute the ultimate transverse Strength of any rectangular Beam, when supported at both Ends and loaded in the Centre.*

Rule I. Multiply the tabular value of s by 4 times the breadth and square of the depth in inches, and divide that product by the length, also in inches, for the weight.

Rule II. 1. Compute the ultimate deflection by Prob. V.; square that deflection, and divide it by the square of half the length of the beam, and add the quotient to 1, for the square of the secant of deflection; which multiply by the length in inches.

2. Multiply the tabular value of s' by 4 times the breadth, and the square of the depth; and divide that product by the former, for the answer in lbs.

EXAM. What weight will be necessary to break a piece of larch similar to the 3rd specimen, the length being 8 feet 4 inches, the breadth 8 inches, and depth 10 inches; being supported at each end, and loaded in the middle?

Ans. 36676lbs.

Note 1. When the beam is loaded uniformly throughout its length, the same rule will apply, but the result must be doubled.

2. If the beam be *fixed* at each end and loaded in the middle, then the result obtained in the problem must be increased by its half.

3. If the beam be fixed at both ends and loaded uniformly

throughout its length, the same result must be multiplied by 3. That is, the strength under these several circumstances:

Supported and loaded in the centre...	are as	1	: 2
Do. and loaded throughout its length		2	: 4
Fixed and loaded in the centre.........		$1\frac{1}{2}$	: 3
Do. loaded throughout its length......		3	: 6

EXAM. 2. A piece of New England fir, 10 feet long, and 6 inches square, being fixed at each end, and loaded uniformly through its entire length: it is required to find the weight necessary to break it. Ans. 24036lbs.

PROB. VIII. *To find the Weight under which a Column of Timber of given Dimensions and Elasticity will begin to bend, when placed, vertically, on a horizontal Plane.*

Rule. Multiply into one sum the value of E for the proposed wood; the cube of the least thickness, and the greatest thickness, the two latter both in inches; and that product again by the constant number ·2056. Then divide the last product by the square of the length, in inches, for the answer, or weight in lbs.*

EXAM. 1. What weight will be requisite to bend a rod of red pine, 10 inches in length and 1 inch square, when placed vertically on a plane, the weight being applied at its upper extremity? Ans. 15131lbs.

* This rule is founded upon the formulæ which have been given for this particular case, by *Euler, Poisson,* &c.

viz. $\text{E}kk = \dfrac{\text{P}f^3}{3b}$ = *absolute elasticity.*

$$\text{Q} = \frac{\pi^2 \text{E}kk}{4f^2} = \frac{\pi^2 \text{P}f}{12b} = \textit{the weight}, \text{ under which a column}$$

begins to bend. Where P is half the weight, f half the length, and b the deflection, when the beam or column is loaded in the middle, and supported at its two ends: also, $\pi = 3{\cdot}14159$, &c. or the semicircumference of a circle to radius 1; that is, according to our notation, $\text{E}kk = \dfrac{\frac{1}{2}\text{W}(\frac{1}{2}l)^3}{3\delta}$, or $\text{E}kk = \dfrac{\text{W}l^3}{48\delta}$, but we have $\text{E} = \dfrac{\text{W}l^3}{ad^3\delta}$; whence $\text{E}kk = \dfrac{\text{E}ad^3}{48}$. And substituting this in our second formula for Ekk, and l for $2f$, we have

$$\text{Q} = \frac{\pi^2 \text{E}ad^3}{48l^2} = \frac{{\cdot}2056\text{E}ad^3}{l^2};$$

which is the same as the rule in words.

EXAM. 2. Assuming the elasticity of English oak at 5806200, what weight will it require to bend a column, 8 feet 4 inches in length and 10 inches square?

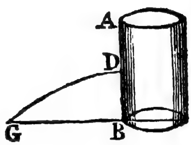
Ans. 1193754lbs.

EXAM. 3. What weight will it require to bend a column of the same wood, and the same lateral dimensions, but of double the length? Ans. 298438lbs.

PRACTICAL QUESTIONS.

QUESTION I.

A LARGE vessel, of 10 feet, or any other given depth, and of any shape, being kept constantly full of water, by means of a supplying cock, at the top; it is proposed to assign the place where a small hole must be made in the side of it, so that the water may spout through it to the greatest distance on the plane of the base.

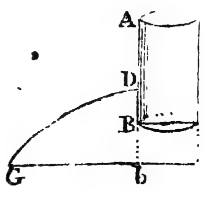


Let AB denote the height or side of the vessel; D the required hole in the side, from which the water spouts, in the parabolic curve DG, to the greatest distance BG, on the horizontal plane.

By the scholium art. 268, Hydraulics, the distance BG is always equal to $2\sqrt{AD \cdot DB}$, which is equal to $2\sqrt{x(a-x)}$ or $2\sqrt{ax-x^2}$, if a be put to denote the whole height AB of the vessel, and $x =$ AD the depth of the hole. Hence $2\sqrt{ax-x^2}$, or $ax - x^2$, must be a maximum. In fluxions, $a\dot{x} - 2x\dot{x} = 0$, or $a - 2x = 0$, and $2x = a$, or $x = \frac{1}{2}a$. So that the hole D must be in the middle between the top and bottom; the same as before found at the end of the scholium above quoted.

QUESTION II.

If the same vessel as in QUEST. 1, stand on high, with its bottom a given height above a horizontal plane below; it is proposed to determine where the small hole must be made, so as to spout farthest on the said plane.



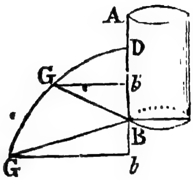
Let the annexed figure represent the vessel as before, and bG the greatest distance spouted by the fluid, DG, on the plane bG.

Here, as before, $b\text{G} = 2\sqrt{\text{AD} \cdot \text{D}b} = 2\sqrt{x(c-x)} = 2\sqrt{cx-x^2}$, by

putting $Ab = c$, and $AD = x$. So that $2\sqrt{cx - x^2}$ or $cx - x^2$ must be a maximum. And hence, like as in the former question, $x = \frac{1}{2}c = \frac{1}{2}Ab$. So that the hole D must be made in the middle between the top of the vessel, and the given plane, that the water may spout farthest.

QUESTION III.

But if the same vessel, as before, stand on the top of an inclined plane, making a given angle, as suppose of 30 degrees, with the horizon; it is proposed to determine the place of the small hole, so as the water may spout the farthest on the said inclined plane.



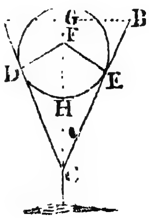
Here again (D being the place of the hole, and BG the given inclined plane), $bG = 2\sqrt{AD \,.\, Db} = 2\sqrt{x(a - x \pm z)}$, putting $z = Bb$, and, as before, $a = AB$, and $x = AD$. Then bG must still be a maximum, as also Bb, being in a given ratio to the maximum BG, on account of the given angle B. Therefore $ax - x^2 \pm xz$, as well as z, is a maximum. Hence, by art. 59 of the Fluxions, $a\dot{x} - 2x\dot{x} \pm z\dot{x} = 0$, or $a - 2x \pm z = 0$; conseq. $\pm z = 2x - a$; and hence $bG = 2\sqrt{x(a - x \pm z)}$ becomes barely $2x$. But as the given angle BGb is $= 30°$, the sine of which is $\frac{1}{2}$; therefore $BG = 2Bb$ or $2z$, and $bG^2 = BG^2 - Bb^2 = 3z^2 = 3(2x - a)^2$, or $bG = \pm(2x - a)\sqrt{3}$.

Putting now, these two values of bG equal to each other, gives the equation $2x = \pm(2x - a)\sqrt{3}$, from which is found

$$x = \frac{\frac{1}{2}a\sqrt{3}}{\sqrt{3} \pm 1} = \frac{3 \pm \sqrt{3}}{4}a,$$ the value of AD required.

QUESTION IV.

It is required to determine the size of a ball, which, being let fall into a conical glass full of water, shall expel the most water possible from the glass; its depth being 6, and diameter 5 inches.



Let ABC represent the cone of the glass, and DHE the ball, touching the sides in the points D and E, the centre of the ball being at some point F in the axis GC of the cone.

Put $AG = GB = 2\frac{1}{4} = a$,
$CG = 6 = b$,
$AC = \sqrt{AG^2 + GC^2} = 6\frac{1}{2} = c$,
$FD = FE = FH = x$ the radius of the ball.

The two triangles ACG and DCF are equiangular; theref.

$AG : AC :: DF : FC$, that is, $a : c :: x : \frac{cx}{a} = FC$; hence

$GF = GC - FC = b - \frac{cx}{a}$, and $GH = GF + FH = b + x - \frac{cx}{a}$,

the height of the segment immersed in the water. Then (by rule 1 for the spherical segment, page 51), the content of the said immersed segment will be $(6DF - 2GH \times GH^2$

$\times \cdot 5236 = (2x - b + \frac{cx}{a}) \times (x + b - \frac{cx}{a})^2 \times 1 \cdot 0472$,

which must be a maximum, by the question; the fluxion of this made $= 0$, and divided by $2\dot{x}$ and the common factors,

gives $\frac{2a+c}{a} \times (b - \frac{c-a}{a}x) - (\frac{2a+c}{a}x - b) \times \frac{c-a}{a} \times 2 = 0$;

this reduced gives $x = \frac{abc}{(c-a) \times (c+2a)} = 2\frac{11}{21}$, the radius of the ball. Consequently its diameter is $4\frac{11}{40}$ inches, as required.

PRACTICAL EXERCISES CONCERNING FORCES; WITH THE RELATION BETWEEN THEM AND THE TIME, VELOCITY, AND SPACE DESCRIBED.

Before entering on the following problems, it will be convenient here to lay down a synopsis of the theorems which express the several relations between any forces, and their corresponding times, velocities, and spaces described; which are all comprehended in the following 12 theorems, as collected from the principles in the foregoing parts of this work.

Let f, F, be any two constant accelerative forces, acting on any body, during the respective times t, T, at the end of which are generated the velocities v, V, and described the spaces s, S. Then, because the spaces are as the times and velocities conjointly, and the velocities as the forces and times; we shall have,

I. *In Constant Forces.*

$$1.\quad \frac{s}{S} = \frac{tv}{TV} = \frac{t^2f}{T^2F} = \frac{v^2F}{V^2f}.$$

$$2.\quad \frac{v}{V} = \frac{ft}{FT} = \frac{sT}{St} = \sqrt{\frac{fs}{FS}}.$$

$$3.\quad \frac{t}{T} = \frac{Fv}{fV} = \frac{sV}{Sv} = \sqrt{\frac{Fs}{fS}}.$$

$$4.\quad \frac{f}{F} = \frac{Tv}{tV} = \frac{T^2s}{t^2S} = \frac{v^2S}{V^2s}.$$

And if one of the forces, as F, be the force of gravity at the surface of the earth, and be called 1, and its time T be $= 1''$; then it is known by experiment that the corresponding space S is $= 16\frac{1}{12}$ feet, and consequently its velocity $V = 2S = 32\frac{1}{6}$, which call g. Then the above four theorems, in this case, become as here below:

$$5.\quad s = \tfrac{1}{2}tv = \tfrac{1}{2}gft^2 = \frac{v^2}{2gf}.$$

$$6.\quad v = \frac{2s}{t} = gft = \sqrt{2gfs}.$$

$$7.\quad t = \frac{2s}{v} = \frac{v}{gf} = \sqrt{\frac{s}{\frac{1}{2}gf}}.$$

$$8.\quad f = \frac{v}{gt} = \frac{2s}{gt^2} = \frac{v^2}{2gs}.$$

And from these are deduced the following four theorems, for variable forces, viz.

II. *In Variable Forces.*

$$9.\quad \dot{s} = v\dot{t} = \frac{v\dot{v}}{gf}.$$

$$10.\quad \dot{v} = gf\dot{t} = \frac{\quad}{v}$$

$$11.\quad \dot{t} = \frac{\dot{s}}{v} = \frac{\dot{v}}{gf}.$$

$$12.\quad f = \frac{v\dot{v}}{g\dot{s}} = \frac{\dot{v}}{g\dot{t}}.$$

In these last four theorems, the force f, though variable, is supposed to be constant for the indefinitely small time t, and they are to be used in all cases of variable forces, as the former ones in constant forces; namely from the circumstances of the problem under consideration, an expression is deduced for the value of the force f, which being substituted in one of these theorems, that may be proper to the case in hand; the equation thence resulting will determine the corresponding values of the other quantities, required in the problem.

When a motive force happens to be concerned in the question, it may be proper to observe, that the motive force m, of a body, is equal to fq, the product of the accelerative force, and the quantity of matter in it q; and the relation between these three quantities being universally expressed by this equation $m = qf$, it follows that, by means of it, any one of the three may be expelled out of the calculation, or else brought into it.

Also, the momentum, or quantity of motion in a moving body, is qv, the product of the velocity and matter.

It is also to be observed, that the theorems equally hold good for the destruction of motion and velocity, by means of retarding forces, as for the generation of the same, by means of accelerating forces.

To the following problems, which are all resolved by the application of these theorems, it has been thought proper to subjoin their solutions, for the better information and convenience of the student.

PROBLEM I.

To determine the time and velocity of a body descending, by the force of gravity, down an inclined plane; the length of the plane being 20 feet, and its height 1 foot.

Here, by Mechanics, the force of gravity being to the force down the plane, as the length of the plane is to its height, therefore as $20 : 1 :: 1$ (the force of gravity) : $\frac{1}{20} = f$, the force on the plane.

Therefore, by theor. 6, v or $\sqrt{2gfs}$ is $\sqrt{(4 \times 16\frac{1}{12} \times \frac{1}{20} \times 20)} = \sqrt{(4 \times 16\frac{1}{12})} = 2 \times 4\frac{1}{96}$ or $8\frac{1}{48}$ feet nearly, the last velocity per second. And,

By theor. 7, t or $\sqrt{\frac{s}{\frac{1}{2}gf}}$ is $\sqrt{\frac{20}{16\frac{1}{12} \times \frac{1}{20}}} = \sqrt{\frac{400}{16\frac{1}{12}}} = \frac{20}{4\frac{1}{96}}$ $= 4\frac{76}{77}$ seconds, the time of descending.

PROBLEM II.

If a cannon ball be fired with a velocity of 1000 feet per second, up a smooth inclined plane, which rises 1 foot in 20: it is proposed to assign the length which it will ascend up the plane, before it stops and begins to return down again, and the time of its ascent.

Here $f = \frac{1}{20}$ as before.

Then, by theor. 5, $s = \frac{v^2}{2gf} = \frac{1000^2}{4 \times 16\frac{1}{12} \times \frac{1}{20}} = \frac{60000000}{193}$ $= 310880\frac{160}{193}$ feet, or nearly 59 miles, the distance moved.

And, by theor. 7, $t = \frac{v}{gf} = \frac{1000}{2 \times 16\frac{1}{12} \times \frac{1}{20}} = \frac{120000}{193}$ $= 621''\frac{147}{193} = 10'\ 21''\frac{147}{193}$, the time of ascent.

PROBLEM III.

If a ball be projected up a smooth inclined plane, which rises 1 foot in 10, and ascend 100 feet before it stop: required the time of ascent, and the velocity of projection.

First, by theor. 6, $v = \sqrt{2gfs} = \sqrt{(4 \times 16\frac{1}{12} \times \frac{1}{10} \times 100)}$ $= 8\frac{1}{48}\sqrt{10} = 25{\cdot}36408$ feet per second, the velocity.

And by theor. 7, $t = \sqrt{\frac{s}{\frac{1}{2}gf}} = \sqrt{\frac{100}{16\frac{1}{12} \times \frac{1}{10}}} = \frac{10}{4\frac{1}{96}}\sqrt{10} =$ $\frac{192}{77}\sqrt{10} = 7{\cdot}88516$ seconds, the time in motion.

PROBLEM IV.

If a ball be observed to ascend up a smooth inclined plane, 100 feet in 10 seconds, before it stop, to return back again: required the velocity of projection, and the angle of the plane's inclination.

First, by theor. 6, $v = \frac{2s}{t} = \frac{200}{10} = 20$ feet per second, the velocity.

And, by theor. 8, $f = \frac{2s}{gt^2} = \frac{2 \cdot 100}{2 \cdot 16\frac{1}{12} \times 100} = \frac{12}{193}$. That is, the length of the plane is to its height, as 193 to 12.

Therefore 193 : 12 : : 100 : 6·2176 the height of the plane, or the sine of elevation to radius 100, which answers to $3^\circ\ 34'$, the angle of elevation of the plane.

PROBLEM V.

By a mean of several experiments, I have found, that a cast iron ball, of 2 inches diameter, fired perpendicularly into the face or end of a block of elm wood, or in the direction of the fibres, with a velocity of 1500 *feet per second, penetrated* 13 *inches deep into its substance. It is proposed then to determine the time of the penetration, and the resisting force of the wood, as compared to the force of gravity, supposing that force to be a constant quantity.*

First, by theor. 7, $t = \frac{2s}{v} = \frac{2 \times 13}{1500 \times 12} = \frac{1}{692}$ part of a second, the time in penetrating.

And, by theor. 8, $f = \frac{v^2}{2gs} = \frac{1500^2}{4 \times 16\frac{1}{12} \times \frac{13}{12}} = \frac{81000000}{13 \times 193}$ $= 32284$. That is, the resisting force of the wood, is to the force of gravity, as 32284 to 1.

But this number will be different, according to the diameter of the ball, and its density or specific gravity. For, since f is as $\frac{v^2}{s}$ by theor. 4, the density and size of the ball remaining the same; if the density, or specific gravity, n, vary, and all the rest be constant, it is evident that f will be as n; and therefore f as $\frac{nv^2}{s}$ when the size of the ball only is constant. But when only the diameter d varies, all the rest being constant, the force of the blow will vary as d^3, or as the magnitude of the ball; and the resisting surface, or force of resistance, varies as d^2; therefore f is as $\frac{d^3}{d^2}$, or as d only, when all the rest are constant. Consequently f is as $\frac{dnv^2}{s}$ when they are all variable.

And so $\frac{f}{F} = \frac{dnv^2 S}{DNV^2 s}$, and $\frac{s}{S} = \frac{dnv^2 F}{DNV^2 f}$; where f denotes the strength or firmness of the substance penetrated, and is here supposed to be the same, for all balls and velocities, in the same substance, which is either accurately or nearly so. See page 214, vol. iii. of my Tracts.

Hence, taking the numbers in the problem, it is - - -

$f = \frac{dnv^2}{s} = \frac{\frac{2}{12} \times 7\frac{1}{3} \times 1500^2}{\frac{13}{12}} = \frac{44 \times 1500^2}{39} = 2538462$ the

value of f for elm wood. Where the specific gravity of the ball is taken $7\frac{1}{3}$, which is a little less than that of solid cast iron, as it ought, on account of the air bubble which is found in all cast balls.

PROBLEM VI.

To find how far a 24lb. ball of cast iron will penetrate into a block of sound elm, when fired with a velocity of 1600 *feet per second.*

Here, because the substance is the same as in the last problem, both of the balls and wood, $N = n$, and $F = f$; therefore $\frac{S}{s} = \frac{DV^2}{dv^2}$, or $S = \frac{DV^2s}{dv^2} = \frac{5.55 \times 1600^2 \times 13}{2 \times 1500^2} = 41\frac{2}{45}$ inches nearly, the penetration required.

PROBLEM VII.

It was found by Mr. Robins (vol. i. p. 273, *of his works*), *that an* 18-*pounder ball, fired with a velocity of* 1200 *feet per second, penetrated* 34 *inches into sound dry oak. It is required then to ascertain the comparative strength or firmness of oak and elm.*

The diameter of an 18lb. ball is 5·04 inches $= D$. Then, by the numbers given in this problem for oak, and in prob. 5, for elm, we have - - - - - - - -

$$\frac{f}{F} = \frac{dv^2S}{DV^2s} = \frac{2 \times 1500^2 \times 34}{5{\cdot}04 \times 1200^2 \times 13} = \frac{100 \times 17}{5{\cdot}04 \times 16 \times 13} = \frac{1700}{1048} \text{ or}$$
$= \frac{8}{5}$ nearly.

From which it would seem, that elm timber resists more than oak, in the ratio of about 8 to 5; which is not probable, as oak is a much firmer and harder wood. But it is to be suspected that the great penetration in Mr. R.'s experiment was owing to the splitting of the timber in some degree.

PROBLEM VIII.

A 24-pounder ball being fired into a bank of firm earth, with a velocity of 1300 *feet per second, penetrated* 15 *feet. It is required then to ascertain the comparative resistances of elm and earth.*

Comparing the numbers here with those in prob. 5, it is

$$\frac{f}{\text{F}} = \frac{dv^2s}{\text{DV}^2s} = \frac{2 \times 1500^2 \times 15 \times 12}{5{\cdot}55 \times 1300^2 \times 13} = \frac{15^2 \times 24}{13^3 \times 0{\cdot}37} = \tfrac{1800}{271} = \tfrac{20}{3}$$

nearly $= 6\frac{2}{3}$ nearly. That is, elm timber resists about $6\frac{2}{3}$ times more than earth.

PROBLEM IX.

To determine how far a leaden bullet, of $\frac{3}{4}$ of an inch diameter, will penetrate dry elm; supposing it fired with a velocity of 1700 *feet per second, and that the lead does not change its figure by the stroke against the wood.*

Here $\text{D} = \frac{3}{4}$, $\text{N} = 11\frac{1}{3}$, $n = 7\frac{1}{3}$. Then, by the numbers and theorem in prob. 5, it is $s =$

$$\frac{\text{DNV}^2s}{dnv^2} = \frac{\frac{3}{4} \times 11\frac{1}{3} \times 1700^2 \times 13}{2 \times 7\frac{1}{3} \times 1500^2} = \frac{17^3 \times 13}{200 \times 33} = \frac{63869}{6600} = 9\tfrac{2}{3}$$

inches nearly, the depth of penetration.

But as Mr. Robins found this penetration, by experiment, to be only 5 inches; it follows, either that his timber must have resisted about twice as much; or else, which is much more probable, that the defect in the penetration arose from the change of figure in the leaden ball he used, from the blow against the wood.

PROBLEM X.

A one pound ball, projected with a velocity of 1500 *feet per second, having been found to penetrate* 13 *inches deep into dry elm: It is required to ascertain the time of passing through every single inch of the* 13, *and the velocity lost at each of them; supposing the resistance of the wood constant or uniform.*

The velocity v being 1500 feet, or $1500 \times 12 = 18000$ inches, and velocities and times being as the roots of the spaces, in constant retarding forces, as well as in accelerating ones, and t being $= \dfrac{2s}{v} = \dfrac{26}{12 \times 1500} = \dfrac{13}{9000} = \dfrac{1}{692}$ part of a second, the whole time of passing through the 13 inches; therefore as

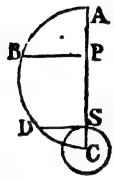
$$\sqrt{13} : \sqrt{13} - \sqrt{12} :: v :$$

veloc. lost		Time in the
$\frac{\sqrt{13}-\sqrt{12}}{\sqrt{13}}v = 58\cdot9 :: t :$	$\frac{\sqrt{13}-\sqrt{12}}{\sqrt{13}}t = \cdot00005$	1st [inch.
$\frac{\sqrt{12}-\sqrt{11}}{\sqrt{13}}v = 61\cdot4 :: t :$	$\frac{\sqrt{12}-\sqrt{11}}{\sqrt{13}}t = \cdot00006$	2d
$\frac{\sqrt{11}-\sqrt{10}}{\sqrt{13}}v = 64\cdot2$, &c.	$\frac{\sqrt{11}-\sqrt{10}}{\sqrt{13}}t = \cdot00006$	3d
$\frac{\sqrt{10}-\sqrt{9}}{\sqrt{13}}v = 67\cdot5$	$\frac{\sqrt{10}-\sqrt{9}}{\sqrt{13}}t = \cdot00007$	4th
$\frac{\sqrt{9}-\sqrt{8}}{\sqrt{13}}v = 71\cdot4$	$\frac{\sqrt{9}-\sqrt{8}}{\sqrt{13}}t = \cdot00007$	5th
$\frac{\sqrt{8}-\sqrt{7}}{\sqrt{13}}v = 76\cdot0$	$\frac{\sqrt{8}-\sqrt{7}}{\sqrt{13}}t = \cdot00007$	6th
$\frac{\sqrt{7}-\sqrt{6}}{\sqrt{13}}v = 81\cdot7$	$\frac{\sqrt{7}-\sqrt{6}}{\sqrt{13}}t = \cdot00008$	7th
$\frac{\sqrt{6}-\sqrt{5}}{\sqrt{13}}v = 88\cdot8$	$\frac{\sqrt{6}-\sqrt{5}}{\sqrt{13}}t = \cdot00008$	8th
$\frac{\sqrt{5}-\sqrt{4}}{\sqrt{13}}v = 98\cdot2$	$\frac{\sqrt{5}-\sqrt{4}}{\sqrt{13}}t = \cdot00009$	9th
$\frac{\sqrt{4}-\sqrt{3}}{\sqrt{13}}v = 111\cdot4$	$\frac{\sqrt{4}-\sqrt{3}}{\sqrt{13}}t = \cdot00011$	10th
$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{13}}v = 132\cdot2$	$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{13}}t = \cdot00013$	11th
$\frac{\sqrt{2}-\sqrt{1}}{\sqrt{13}}v = 172\cdot3$	$\frac{\sqrt{2}-\sqrt{1}}{\sqrt{13}}t = \cdot00017$	12th
$\frac{\sqrt{1}-\sqrt{0}}{\sqrt{13}}v = 416\cdot0$	$\frac{\sqrt{1}-\sqrt{0}}{\sqrt{13}}t = \cdot00040$	13th
Sum $1500\cdot0$	Sum $\frac{1}{692}$ or $\cdot00144$	

Hence, as the motion lost at the beginning is very small; and consequently the motion communicated to any body, as an inch plank, in passing through it, is very small also; we can conceive how such a plank may be shot through, when standing upright, without oversetting it.

PROBLEM XI.

The force of attraction, above the earth, being inversely as the square of the distance from the centre; it is proposed to determine the time, velocity, and other circumstances, attending a heavy body falling from any given height; the descent at the earth's surface being $16\frac{1}{12}$ feet, or 193 inches, in the first second of time.



Put

$r =$ CS the radius of the earth,
$a =$ CA the dist. fallen from,
$x =$ CP any variable distance,
$v =$ the velocity at P,
$t =$ time of falling there, and
$\frac{1}{2}g = 16\frac{1}{12}$, half the veloc. or force at S,
$f =$ the force at the point P.

Then we have the three following equations, viz.

$x^2 : r^2 :: 1 : f = \frac{r^2}{x^2}$ the force at P, when the force of gravity at the surface is considered as 1;

$t\dot{v} = -\dot{x}$, because x decreases; and

$v\dot{v} = -gf\dot{x} = -\frac{gr^2\dot{x}}{x^2}$.

The fluents of the last equation give $v^2 = \frac{2gr^2}{x}$. But when $x = a$, the velocity $v = 0$; therefore, by correction,

$v^2 = \frac{2gr^2}{x} - \frac{2gr^2}{a} = 2gr^2 \times \frac{a-x}{ax}$; or $v = \sqrt{(\frac{2gr^2}{a} \times \frac{a-x}{x})}$, a general expression for the velocity at any point P.

When $x = r$, this gives $v = \sqrt{(2gr \times \frac{a-r}{a})}$ for the greatest velocity, or the velocity when the body strikes the earth.

When a is very great in respect of r, the last velocity becomes $(1 - \frac{r}{2a}) \times \sqrt{2gr}$ very nearly, or nearly $\sqrt{2gr}$ only, which is accurately the greatest velocity by falling from an infinite height. And this, when $r = 3965$ miles, is $6{\cdot}9506$ miles per second. Also, the velocity acquired in falling from

the distance of the sun, or 12000 diameters of the earth, is 6·9505 miles per second. And the velocity acquired in falling from the distance of the moon, or 30 diameters, is 6·8927 miles per second.

Again, to find the time; since $\dot{t}v = -\dot{x}$, therefore $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{a}{2gr^2}} \times \frac{-x\dot{x}}{\sqrt{ax - xx}}$; the correct fluent of which gives $t = \sqrt{\frac{a}{2gr^2}} \times (\sqrt{ax - xx}$ + arc to diameter a and vers. $a - x)$; or the time of falling to any point P $= \frac{1}{2r}\sqrt{\frac{a}{\frac{1}{2}g}} \times (\text{AB} + \text{BP})$. And when $x = r$, this becomes $t = \frac{1}{2}\sqrt{\frac{a}{\frac{1}{2}g}} \times \frac{\text{AD} + \text{DS}}{\text{SC}}$ for the whole time of falling to the surface at S; which is evidently infinite when a or AC is infinite, though the velocity is then only the finite quantity $\sqrt{2gr}$.

When the height above the earth's surface is given $= g$; because r is then nearly $= a$, and AD nearly $=$ DS, the time t for the distance g will be nearly - - - - - - - - - -

$$\sqrt{\frac{1}{2gr^2}} \times 2\text{DS} = \sqrt{\frac{1}{2gr}} \times \sqrt{2gr} = 1'', \text{ as it ought to be.}$$

If a body, at the distance of the moon at A, fall to the earth's surface at S. Then $r = 3965$ miles, $a = 60r$, and $t = 416806'' = 4$ da. 19 h. 46 m. 46 s. which is the time of falling from the moon to the earth.

In like manner the time of falling from the distance of the sun would be 64 da. 13 h. 46 m. 46 s.

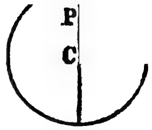
When the attracting body is considered as a point C; the whole time of descending to C will be - - - - - - - -

$$\frac{1}{2r}\sqrt{\frac{a}{\frac{1}{2}g}} \times \text{ABCD} = \frac{\cdot 7854a}{r}\sqrt{\frac{a}{\frac{1}{2}g}} = \frac{10a}{51r}\sqrt{a} = \frac{\cdot 7854}{r}\sqrt{\frac{a^3}{\frac{1}{2}g}}.$$

Hence, the times employed by bodies, in falling from quiescence to the centre of attraction, are as the square roots of the cubes of the heights from which they respectively fall.

PROBLEM XII.

The force of attraction below the earth's surface being directly as the distance from the centre: it is proposed to determine the circumstances of velocity, time, and space fallen by a heavy body from the surface, through a perforation made straight to the centre of the earth: abstracting from the effect of the earth's rotation, and supposing it to be a homogeneous sphere 3965 miles radius.



Put $r = $ AC the radius of the earth,
$x = $ CP the dist. from the centre,
$v = $ the velocity at P,
$t = $ the time there,
$\frac{1}{2}g = 16\frac{1}{12}$, half the force at A,
$f = $ the force at P.

Then CA : CP :: $1 : f$; and the three equations are $rf = x$, and $v\dot{v} = -gf\dot{x}$, and $\dot{t}v = -\dot{x}$.

Hence $f = \frac{x}{r}$, and $v\dot{v} = \frac{-gx\dot{x}}{r}$; the correct fluent of which gives $v = \sqrt{(g \times \frac{r^2 - x^2}{r})} = \text{PD}\sqrt{\frac{g}{r}} = \text{PD}\sqrt{\frac{g}{\text{CE}}}$, the velocity at the point P; where PD and CE are perpendicular to CA. So that the velocity at any point P, is as the perpendicular or sine PD at that point.

When the body arrives at C, then $v = \sqrt{gr} = \sqrt{g \cdot \text{AC}}$ $= 25950$ feet or $4 \cdot 9148$ miles per second, which is the greatest velocity, or that at the centre C.

Again, for the time; $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{r}{g}} \times \frac{-\dot{x}}{\sqrt{r^2 - x^2}}$; and the fluents give $t = \sqrt{\frac{r}{g}} \times$ arc to cosine $\frac{x}{r} = \sqrt{\frac{1}{gr}} \times$ arc AD. So that the time of descent to any point P, is as the corresponding arc AD.

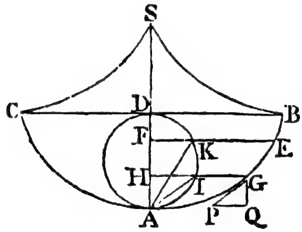
When P arrives at C, the above becomes $t = $ - - - - $\sqrt{\frac{1}{gr}} \times$ quadrant AB $= \frac{\text{AB}}{\text{AC}}\sqrt{\frac{r}{g}} = 1 \cdot 5708\sqrt{\frac{r}{g}} = 1267\frac{1}{4}$ seconds $= 21$ m. $7\frac{1}{4}$ s. for the time of falling to the centre C.

The time of falling to the centre is the same quantity $1 \cdot 5708\sqrt{\frac{r}{g}}$, from whatever point in the radius AC the body begins to move. For, let n be any given distance from C at which the motion commences: then by correction, $v = \sqrt{(\frac{g}{r} \cdot \overline{n^2 - x^2})}$, and hence $\dot{t} = \sqrt{\frac{r}{g}} \times \frac{-\dot{x}}{\sqrt{n^2 - x^2}}$, the fluents of which give $t = \sqrt{\frac{r}{g}} \times$ arc to cosine $\frac{x}{n}$; which, when $x = 0$, gives $t = \sqrt{\frac{r}{g}} \times$ quadrant $= 1 \cdot 5708\sqrt{\frac{r}{g}}$, for the time of descent to the centre C, the same as before.

As an equal force, acting in contrary directions, generates or destroys an equal quantity of motion, in the same time; it follows that, after passing the centre, the body will just ascend to the opposite surface at B, in the same time in which it fell to the centre from A. Then from B it will return again in the same manner, through C to A; and so oscillate continually between A and B, the velocity being always equal at equal distances from C on both sides; and the whole time of a double oscillation, or of passing from A and arriving at A again, will be quadruple the time of passing over the radius AC, or $= 2 \times 3{\cdot}1416 \sqrt{\frac{r}{g}} =$ 1h. 24m. 29s.

PROBLEM XIII.

To find the Time of a Pendulum vibrating in the Arc of a Cycloid.



Let S be the point of suspension;
SA the length of pendulum;
CAB, the whole cycloidal arc;
AIKD, the generating circle, to which FKE, HIG are perpendiculars.
SC, SB two other equal semicloids, on which the thread wrapping, the end A is made to describe the cycloid BAC.

By the nature of the cycloid, AD = DS; and SA = 2AD = SC = SB = CA = AB. Also, if at any point G be drawn the tangent GP; GQ parallel and PQ perpendicular to AD: then PG is parallel to the chord AI, by the nature of the curve. And, by the nature of forces, the force of gravity: force in direction GP : : GP : GQ : : AI : AH : : AD : AI; in like manner, the force of gravity: force in the curve at E : : AD : AK; that is, the accelerative force in the curve, is every where as the corresponding chord AI or AK of the circle, or as the arc AG or AE of the cycloid, since AG is always = 2AI, by the nature of the curve. So that the process and conclusions, for the velocity and time of describing any arc in this case, will be the very same as in the last problem, the nature of the forces being the same, viz. as the distance to be passed over to the lowest point A.

From which it follows, that the time of a semi-vibration, in all arcs, AG, AE, &c, is the same constant quantity $1\cdot5708\sqrt{\frac{r}{g}} = 1\cdot5708\sqrt{\frac{AS}{g}} = 1\cdot5708\sqrt{\frac{l}{g}}$; and the time of a whole vibration from B to C, or from C to B, is $3\cdot1416\sqrt{\frac{l}{g}}$; where l = AS = AB is the length of the pendulum, and 3·1416 the circumference of a circle whose diameter is 1.

Since the time of a body's falling by gravity through $\frac{1}{2}l$, or half the length of the pendulum, by the nature of descents, is $\sqrt{\frac{l}{g}}$, which being in proportion to $3\cdot1416\sqrt{\frac{l}{g}}$, as 1 is to 3·1416; therefore the diameter of a circle is to its circumference, as the time of falling through half the length of a pendulum, is to the time of one vibration.

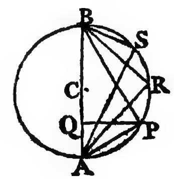
If the time of the whole vibration be 1 second, this equation arises, viz. $1'' = 3\cdot1416\sqrt{\frac{l}{g}}$; hence $l = \frac{g}{3\cdot1416^2} = \frac{\frac{1}{2}g}{4\cdot9348}$, and $\frac{1}{2}g = 3\cdot1416^2 \times \frac{1}{2}l = 4\cdot9348l$. So that if one of these, g or l, be given by experiment, these equations will give the other. See pages 196, and 222.

Hence the times of vibration of pendulums, are as the square roots of their lengths; and the number of vibrations made in a given time, is reciprocally as the square roots of the lengths. And hence also, the length of a pendulum vibrating n times in a minute, or 60″, is $l = 39\frac{1}{8} \times \frac{60^2}{n^2} = \frac{140850}{nm}$; as at page 232.

When a pendulum vibrates in a circular arc; as the length of the string is constantly the same, the time of vibration will be longer than in a cycloid; but the two times will approach nearer together as the circular arc is smaller; so that when it is very small, the times of vibration will be nearly equal. And hence it happens that $39\frac{1}{8}$ inches is the length of a pendulum vibrating seconds, in the very small arc of a circle.

PROBLEM XIV.

To determine the Time of a Body descending down the Chord of a Circle.



Let C be the centre; AB the vertical diameter; AP any chord, down which a body is to descend from P to A; and PQ perpendicular to AB.

Now, as the natural force of gravity in the vertical direction BA, is to the force

urging the body down the plane PA, as the length of the plane AP, is to its height AQ; therefore the velocity in PA and QA, will be equal at all equal perpendicular distances below PQ; and consequently the - - - - - - - - - - -
time in PA : time in QA : : PA : QA : : BA : PA; but time in BA : time in QA : : $\sqrt{BA}$: $\sqrt{QA}$: : $\sqrt{(BA . BA)}$: $\sqrt{(QA . BA)}$: : BA : PA; hence, as three of the terms in each proportion are the same, the fourth terms must be equal, namely the time in BA = the time PA.

And, in like manner, the time in BP = the time in BA. So that, in general, the times of descending down all the chords BA, BP, BR, BS, &c, of PA, RA, SA, &c, are all equal, and each equal to the time of falling freely through the diameter; as before found at art. 194, Dynamics. Which time is $\sqrt{\frac{2r}{\frac{1}{2}g}}$, where $\frac{1}{2}g = 16\frac{1}{12}$ feet, and r = the radius AC; for $\sqrt{\frac{1}{2}g} : \sqrt{2r} :: 1'' : \sqrt{\frac{2r}{\frac{1}{2}g}}$.

PROBLEM XV.

To determine the Time of filling the Ditches of a Work with Water, at the Top, by a Sluice of 2 Feet square; the Head of Water above the Sluice being 10 Feet, and the Dimensions of the Ditch being 20 Feet wide at Bottom, 22 at Top, 9 deep, and 1000 Feet long.

The capacity of the ditch is 189000 cubic feet.

But $\sqrt{\frac{1}{2}g} : \sqrt{10} :: g : 2\sqrt{5g}$ the velocity of the water through the sluice, the area of which is 4 square feet; therefore $8\sqrt{5g}$ is the quantity per second running through it; and consequently $8\sqrt{5g} : 189000 :: 1'' : \frac{23625}{\sqrt{5g}} = 1863''$ or 31 m. 3 s. nearly, which is the time of filling the ditch.

PROBLEM XVI.

To determine the Time of emptying a Vessel of Water by a Sluice in the Bottom of it, or in the Side near the Bottom: the Height of the Aperture being very small in respect of the Altitude of the Fluid.

Put a = the area of the aperture or sluice;
$g = 32\frac{1}{6}$ feet, the force of gravity;

$d =$ the whole depth of water;
$x =$ the variable altitude of the surface above the aperture;
A $=$ the area of the surface of the water.

Then $\sqrt{\frac{1}{2}g} : \sqrt{x} :: 2g : 2\sqrt{\frac{1}{2}gx}$ the velocity with which the fluid will issue at the sluice; and hence $A : a :: 2\sqrt{\frac{1}{2}gx} : \frac{2a\sqrt{\frac{1}{2}gx}}{A}$, the velocity with which the surface of the water will descend at the altitude x, or the space it would descend in 1 second with the velocity there. Now, in descending the space $\dot{x}$, the velocity may be considered as uniform; and uniform descents are as their times; therefore $\frac{2a\sqrt{\frac{1}{2}gx}}{A} : -\dot{x} :: 1'' : \frac{-A\dot{x}}{2a\sqrt{\frac{1}{2}gx}}$ the time of descending $\dot{x}$ space, or the fluxion of the time of exhausting. That is, $\dot{t} = \frac{-A\dot{x}}{2a\sqrt{\frac{1}{2}gx}}$; which is made negative, because x is a decreasing quantity, or its fluxion negative.

Now, when the nature or figure of the vessel is given, the area A will be given in terms of x; which value of A being substituted into this fluxion of the time, the fluent of the result will be the time of exhausting sought.

So if, for example, the vessel be any prism, or everywhere of the same breadth; then A is a constant quantity, and therefore the fluent is $-\frac{A}{a}\sqrt{\frac{x}{\frac{1}{2}g}}$. But when $x = d$, this becomes $-\frac{A}{a}\sqrt{\frac{d}{\frac{1}{2}g}}$, and should be 0; therefore the correct fluent is $t = \frac{A}{a} \times \frac{\sqrt{d} - \sqrt{x}}{\sqrt{\frac{1}{2}g}}$ for the time of the surface descending till the depth of the water be x. And when $x = 0$, the whole time of exhausting is barely $\frac{A}{a}\sqrt{\frac{d}{\frac{1}{2}g}}$. *

Hence, if A be $= 10000$ square feet, $a = 1$ square foot, and $d = 10$ feet; the time is $7885\frac{1}{5}$ seconds, or 2 h. 11 m. $25\frac{1}{5}$ s.

Again, if the vessel be a ditch, or canal, of 20 feet broad at the bottom, 22 at the top, 9 deep, and 1000 feet long;

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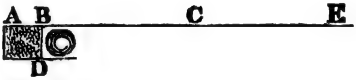
* This last result is obviously inconsistent with the result at pa. 369, vol. iii. The reason is that the supposition of $x = 0$, is incompatible with the hypothesis that the height of the aperture is very small compared with x. The rest of the solution is correct.

then is $90 : 90 + x :: 20 : \frac{90+x}{9} \times 2$ the breadth of the surface of the water when its depth in the canal is x; and therefore $A = \frac{90+x}{9} \times 2000$ is the surface at that time.

Consequently $\dot{t}$ or $\frac{-A\dot{x}}{2a\sqrt{\frac{1}{2}gx}} = 1000 \times \frac{90+x}{9} \times \frac{-\dot{x}}{a\sqrt{\frac{1}{2}gx}}$ is the fluxion of the time; the correct fluent of which, when $x = 0$, is $1000 \times \frac{180+\frac{2}{3}d}{9a} \times \sqrt{\frac{d}{\frac{1}{2}g}} = \frac{1000 \times 186 \times 3}{9 \times 4\frac{1}{96}} =$ $15459''\frac{2}{3}$ nearly, or 4h. 17m. $39\frac{2}{3}$s., being the whole time of exhausting by a sluice of 1 foot square.

PROBLEM XVII.

To determine the Velocity with which a Ball is discharged from a given Piece of Ordnance, with a given Charge of Gunpowder.



Let the annexed figure represent the bore of the gun; AD being the part filled with gunpowder.

And put

a = AB, the part at first filled with powder and the bag;
b = AE, the whole length of the gunbore;
c = ·7854, the area of a circle whose diameter is 1;
d = BD, the diameter of the ball;
e = the specific gravity of the ball, or weight of 1 cubic foot;
$\frac{1}{2}g = 16\frac{1}{12}$ feet, descended by a body in 1 second;
m = 240oz, 15lb, the pressure of the atmosphere on a sq. inch;
n to 1 the ratio of the first force of the fired powder, to the pressure of the atmosphere;
w = the weight of the ball. Also, let
x = AC, be any variable distance of the ball from A, in moving along the gunbarrel.

First, cd^2 is = the area of the circle BD of the ball; theref. mcd^2 is the pressure of the atmosphere on BD; conseq. $mncd^2$ is the first force of the powder on BD.

But the force of the inflamed powder is proportional to its density, and the density is inversely as the space it fills; therefore the force of the powder on the ball at B, is to the force on the same at C, as AC is to AB; that is, - - - -

$x : a :: mncd^2 : \frac{mnacd^2}{x} = F$, the motive force at C:

conseq. $\frac{F}{w} = \frac{mnacd^2}{wx} = f$, the accelerating force there.

Hence, theor. 10 of forces gives $v\dot{v} = gf\dot{x} = \frac{gmnacd^2}{w} \times \frac{\dot{x}}{x}$;

the fluent of which is $v^2 = \frac{2gmnacd^2}{w} \times$ hyp. log. of x.

But when $v = 0$, then $x = a$; therefore by correction,

$v^2 = \frac{2gmnacd^2}{w} \times$ hyp. log. $\frac{x}{a}$ is the correct fluent; conseq.

$v = \sqrt{(\frac{2gmnacd^2}{w} \times \text{hyp. log. } \frac{x}{a})}$ is the veloc. of the ball at C,

and $v = \sqrt{(\frac{2gmnhcd^2}{w} \times \text{hyp. log. } \frac{b}{a})}$ the velocity with which the ball issues from the muzzle at E; where h denotes the length of the cylinder filled with powder; and a the length to the hinder part of the ball, which will be more than h when the ball does not touch the powder.

Or, by substituting the numbers for g, m, c, and changing the hyperbolic logarithms for the common ones, then

$v = \sqrt{(\frac{2306nhd^2}{w} \times \text{com. log. } \frac{b}{a})}$, the velocity at E, in feet.

But, the content of the ball being $\frac{2}{3}cd^3$, its weight is - -

$w = \frac{\frac{2}{3}cd^3e}{12^3} = \frac{ced^3}{2592} = \frac{ed^3}{3300}$; which being substituted for w, in the value of v, it becomes

$v = 2758\sqrt{(\frac{nh}{de} \times \text{com. log. } \frac{b}{a})}$, the velocity at E.

When the ball is of cast iron; taking $e = 7368 = 10(27 \cdot 14)^2$,

the rule becomes $v = 32 \sqrt{(\frac{nh}{d} \times \text{log. } \frac{b}{a})}$ for the veloc. of the cast iron ball.

Or, when the ball is of lead; then $c = 11325 = 10(33 \cdot 652)^2$,

and $v = 26 \sqrt{(\frac{nh}{d} \times \text{log. } \frac{b}{a})}$ for the veloc. of the leaden ball*.

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* Some practical artillerists having expressed a wish to the Editor, to see a solution to this problem upon the supposition *that the gunpowder explodes gradually, so as to be all ignited*

Corol. From the general expression for the velocity v, above given, may be derived what must be the length of the charge of powder a, in the gun-barrel, so as to produce the greatest possible velocity in the ball; namely, by making the

when the ball quits the mouth of the gun, he avails himself of the opportunity of giving it in this place.

Here we must first investigate the relation between the time and the space described. By the hypothesis we shall have the force as $\frac{t}{x}$; and we have, x varying as t^n to find n. We have, from the primitive formulæ, pa. 370. $v\dot{v} = gf\dot{x} \propto f\dot{x}$, and $f \propto \frac{t}{x} \propto \frac{x^{\frac{1}{n}}}{x} \propto \frac{1}{x^{\frac{n-1}{n}}}$. Therefore $v\dot{v} \propto \frac{\dot{x}}{x^{\frac{n-1}{n}}}$; and the fluent

$v^2 \propto nx^{\frac{1}{n}}$. Farther,

$\dot{x} \propto v\dot{t} \propto v \cdot \frac{1}{n} x^{\frac{1}{n}-1} \dot{x} \propto x^{\frac{1}{2n}} \times x^{\frac{1}{n}-1} \dot{x} \propto x^{\frac{3-2n}{2n}} \dot{x}$; so that the fluent $x \propto x^{\frac{3}{2n}}$. Conseq. $1 = \frac{3}{2n}$, and $n = \frac{3}{2}$. That is, in this case, $x \propto t^{\frac{3}{2}}$.

Now, taking the notation of the problem in the text, we have $f' = \frac{mncd^2}{w}$, the accelerative force at the first instant of explosion; and, by hyp. at E, where the whole is exploded, the force will be $\frac{af'}{b}$. To find the force at C, if T be put for the whole time of explosion, we shall have

$$\frac{T}{b}\left(\frac{\text{time}}{\text{space}}\right) : \frac{af'}{b} \text{ (force at E)} :: \frac{t}{x}\left(\frac{\text{time}}{\text{space}}\right) : \frac{af't}{Tx},$$

force at C, or the value of f there, in the general formulæ. Hence we have $v\dot{v} = gf\dot{x} = \frac{gaf't\dot{x}}{Tx}$

But $b : T^{\frac{2}{3}} :: x : t^{\frac{2}{3}} = \frac{\ ^{a}}{b}$ $\therefore t = \frac{Tx^{\frac{3}{2}}}{b^{\frac{3}{2}}}$; and, by substituting this value of t for it, we have

$$v\dot{v} = \frac{gaf'}{Tx} \cdot \frac{Tx^{\frac{3}{2}}\dot{x}}{b^{\frac{3}{2}}} = \frac{gaf'x^{-\frac{1}{2}}\dot{x}}{\ }$$

Taking the fluents, we have

value of v a maximum, or, by squaring and omitting the constant quantities, the expression $a \times$ hyp. log. of $\frac{b}{a}$ a maximum, or its fluxion equal to nothing; that is $\dot{a} \times$ hyp. log. $\frac{b}{a} - \dot{a} = 0$, or hyp. log. of $\frac{b}{a} = 1$; hence $\frac{b}{a} = 2{\cdot}71828$, the number whose hyp. log. is 1. So that $a : b :: 1 : 2{\cdot}71828$, or as 4 to 11 nearly, or nearer as 7 to 19; that is, the length of the charge, to produce the greatest velocity, is the $\frac{4}{11}$th part of the length of the bore, or nearer $\frac{7}{19}$ of it.

By actual experiment it is found, that the charge for the greatest velocity, is but little less than that which is here computed from theory; as may be seen by turning to page 213, vol. 3, of the Tracts, where the corresponding parts are

$$\tfrac{1}{2} v^2 = \frac{gaf'x^{\frac{2}{3}}}{\frac{2}{3}b^{\frac{2}{3}}}, \text{ whence } v = \sqrt{\frac{3gaf'x^{\frac{2}{3}}}{b^{\frac{2}{3}}}}.$$

When x becomes $= b$, this becomes

$$v = \sqrt{(3gaf')} = \sqrt{(3gmnc \cdot \frac{ad^2}{w})}.$$

Cor. 1. Hence, so long as a and d remain the same, the velocity at the muzzle E will be the same whatever be the length of the gun; for b does not appear in the ultimate value of v.

This is contrary to all experience, and proves that the hypothesis is untenable.

Cor. 2. The powder being the same, the velocity at the muzzle (d remaining the same) will be as the square root of the charge.

Cor. 3. In guns of different bores, the velocity at the muzzle will be as $\sqrt{\frac{a}{d}}$. For $v \propto \sqrt{\frac{ad^2}{w}} \propto \sqrt{\frac{ad^2}{d^3}} \propto \sqrt{\frac{a}{d}}$.

Cor. 4. If the charge be given, a will be inversely as d^2, and $v \propto \sqrt{\frac{1}{d^3}}$.

Cor. 5. If b be the length of a gun in which the charge of powder will be all fired when the ball reaches the muzzle, then in a shorter gun AC, the same powder, and an equal charge will give an ultimate velocity varying as $\sqrt{\frac{x^{\frac{2}{3}}}{b^{\frac{2}{3}}}}$ or as $\sqrt[3]{\frac{x}{b}}$.

found to be, for four different lengths of gun, thus, $\frac{3}{10}$, $\frac{3}{12}$, $\frac{3}{16}$, $\frac{3}{20}$; the parts here varying, as the gun is longer, which allows time for the greater quantity of powder to be fired, before the ball is out of the bore.

SCHOLIUM.

In the calculation of the foregoing problem, the value of the constant quantity n remains to be determined. It denotes the first strength or force of the fired gunpowder, just before the ball is moved out of its place. This value is assumed, by Mr. Robins, equal to 1000, that is, 1000 times the pressure of the atmosphere, on any equal spaces.

But the value of the quantity n may be derived much more accurately, from the experiments related in my Tracts, by comparing the velocities there found by experiment, with the rule for the value of v, or the velocity, as computed by theory, viz. - - - - - -

$$v = 100\sqrt{(\frac{na}{10d} \times \text{log.of}\frac{b}{a})}, \text{ or} = 100\sqrt{(\frac{nh}{10d} \times \text{log. of}\frac{b}{a})}.$$

Now, supposing that v is a given quantity, as well as all the other quantities, excepting only the number n, then by reducing this equation, the value of the letter n is found to be as follows, viz. - - - - - - - -

$$n = \frac{dvv}{1000a} \div \text{com. log. of}\frac{b}{a}, \text{ or} = \frac{dvv}{1000h} \div \text{log. of}\frac{b}{a},$$

when h is different from a.

Now, to apply this to the experiments. By pa. 69, vol. 3, of the Tracts, the velocity of the ball, of 1·96 inches diameter, with 4 ounces of powder, in the gun No. 1, was 1100 feet per second; and, by pa. 316, vol. 2, the length of the gun, when corrected for the spheroidal hollow in the bottom of the bore, was 28·53; also, by pa. 48, vol. 3, the length of the charge, when corrected in like manner, was 3·45 inches of powder and bag together, but 2·54 of powder only: so that the values of the quantities in the rule, are thus: $a = 3{\cdot}45$; $b = 28{\cdot}53$; $d = 1{\cdot}96$; $h = 2{\cdot}54$; and $v = 1100$: then, by substituting these values instead of the letters, in the theorem $n = \frac{dvv}{1000a} \div \text{com. log. of}\frac{b}{a}$, it comes out $n = 750$, when h is considered as the same as a. And so on, for the other experiments there treated of.

It is here to be noted however, that there is a circumstance in the experiments delivered in the Tracts, just mentioned, which will alter the value of the letter a in this theorem, which is this, viz. that a denotes the distance of the shot from the bottom of the bore; and the length of the charge of powder alone ought to be the same thing; but, in the experiments, that length included, besides the length of real powder, the substance of the thin flannel bag in which it was always contained, of which the neck at least extended a considerable length, being the part where the open end was wrapped and tied close round with a thread. This circumstance causes the value of n, as found by the theorem above, to come out less than it ought to be, for it shows the strength of the inflamed powder when just fired, and when the flame fills the whole space a before occupied both by the real powder and the bag, whereas it ought to show the first strength of the flame when it is supposed to be contained in the space only occupied by the powder alone, without the bag. The formula will therefore bring out the value of n too little, in proportion as the real space filled by the powder is less than the space filled both by the powder and its bag. In the same proportion therefore must we increase the formula, that is, in the proportion of h, the length of real powder, to a the length of powder and bag together. When the theorem is so corrected, it becomes $\frac{dvv}{1000h} \div$ com. log. of $\frac{b}{a}$.

Now, by pa. 48 and 49, vol. 3, Tracts, there are given both the lengths of all the charges, or values of a, including the bag, and also the length of the neck and bottom of the bag, which is 0·91 of an inch, which therefore must be subtracted from all the values of a, to give the corresponding values of h. This in the example above reduces 3·45 to 2·54.

Hence, by increasing the above result 750, in proportion of 2·54 to 3·45, it becomes 1018. And so on for the other experiments.

But it will be best to arrange the results in a table, with the several dimensions, when corrected, from which they are computed, as follows.

Table of Velocities of Balls and First Force of Powder, &c.

Gun.		Charge of Powder.			Velocity or value of v.	First force, or value of n.
No.	Length, or value of b	Weight in ounces.	Length or value of a.	of h.		
	inches.					
1	28·53	4	3·45	2·54	1100	1018
		8	5·99	5·08	1340	1091
		16	11·07	10·16	1430	967
2	38·43	4	3·45	2·54	1180	1077
		8	5·99	5·08	1580	1193
		16	11·07	10·16	1660	984
3	57·70	4	3·45	2·54	1300	1067
		8	5·99	5·08	1790	1256
		16	11·07	10·16	2000	1076
4	80·23	4	3·45	2·54	1370	1060
		8	5·99	5·08	1940	1289
		16	11·07	10·16	2200	1085

Where it may be observed, that the numbers in the column of velocities, 1430 and 2200, are a little increased, as, from a view of the table of experiments, they evidently required to be. Also the value of the letter d is constantly 1·96 inch.

Hence it appears, that the value of the letter n, used in the theorem, though not yet greatly different from the number 1000, assumed by Mr. Robins, is rather various, both for the different lengths of the gun, and for the different charges with the same gun.

But this diversity in the value of the quantity n, or the first force of the inflamed gunpowder, is probably owing in some measure to the omission of a material datum in the calculation of the problem, namely, the weight of the charge of powder, which has not at all been brought into the computation. For it is manifest, that the elastic fluid has not only the ball to move and impel before it, but its own weight of matter also. The computation may therefore be renewed, in the ensuing problem, to take that datum into the account.

PROBLEM XVIII.

To determine the same as in the last Problem; taking both the Weight of Powder and the Ball into the Calculation.

Besides the notation used in the last problem, let $2p$ denote the weight of the powder in the charge, with the flannel bag in which it was inclosed.

Now, because the inflamed powder occupies at all times the part of the gun bore which is behind the ball, its centre of gravity, or the middle part of the same, will move with only half the velocity that the ball moves with; and this will require the same force as half the weight of the powder, &c, moved with the whole velocity of the ball. Therefore, in the conclusion derived in the last problem, we are now, instead of w, to substitute the quantity $p + w$; and when that is done, the last velocity will come out, $v = \sqrt{(\frac{2230nhd^2}{p + w} \times \text{com. log. } \frac{b}{a})}$.

And from this equation is found the value of n, which is $n = \frac{p + w}{2230hd^2}v^2 \div \text{log. of } \frac{b}{a}, = \frac{p + w}{8567h}v^2 \div \text{log. of } \frac{b}{a}$, by substituting for d its value 1·96, the diameter of the ball.

Now as to the ball, its medium weight was 16 oz. 13 dr. = 16·81 oz. And the weights of the bags containing the several charges of powder, viz. 4 oz., 8 oz., 16 oz., were 8 dr., 12 dr., and 1 oz., 5 dr.; then, adding these to the respective contained weights of powder, the sums, 4·5 oz., 8·75 oz., 17·31 oz., are the values of $2p$, or the weights of the powder and bags; the halves of which, or 2·25, and 4·38, and 8·66, are the values of the quantity p for those three charges; and these being added to 16·81, the constant weight of the ball, there are obtained the three values of $p + w$ for the three charges of powder, which values therefore are 19·06 oz., and 21·19 oz., and 25·47 oz. Then, by calculating the values of the first force n, by the last rule above, with these new data, the whole will be found as in the following table.

The Gun.		Charge of Powder.			Weight of ball and charge, or values of $p+w$.	Velocity, or the values of v.	First force or the value of n.
No.	Length or value of b.	Weight in ounces.	Length or value of a.	Length or value of h.			
	Inches.						
1	28·53	4	3·45	2·54	19·06	1100	1155
		8	5·99	5·08	21·19	1340	1377
		16	11·07	10·16	25·47	1430	1456
2	38·43	4	3·45	2·54	19·06	1180	1167
		8	5·99	5·08	21·19	1580	1506
		16	11·07	10·16	25·47	1660	1492
3	57·70	4	3·45	2·54	19·06	1300	1210
		8	5·99	5·08	21·19	1790	1586
		16	11·07	10·16	25·47	2000	1646
4	80·23	4	3·45	2·54	19·06	1370	1203
		8	5·99	5·08	21·19	1940	1627
		16	11·07	10·16	25·47	2200	1648

And here it appears that the values of n, the first force of the charge, are much more uniform and regular than by the former calculations in the preceding problem, at least in all excepting the smallest charge, 4 oz. in each gun; which it would seem must be owing to some general cause or causes. Nor have we long to search, to find out what those causes may be. For when it is considered that these numbers for the value of n, in the last column of the table, ought to exhibit the first force of the fired powder, when it is supposed to occupy the space only in which the bare powder itself lies; and that whereas it is manifest that the condensed fluid of the charge in these experiments occupies the whole space between the ball and the bottom of the gun bore, or the whole space taken up by the powder and the bag or cartridge together, which exceeds the former space, or that of the powder alone, at least in the proportion of the circle of the gun bore, to the same as diminished by the thickness of the surrounding flannel of the bag that contained the powder; it is manifest that the force was diminished on that account. Now by gently compressing a number of folds of the flannel together, it has been found that the thickness of the single flannel was equal to the 40th part of an inch; the double of which, $\frac{1}{20}$ or ·05 of an inch, is therefore the quantity by which the diameter of the circle of the powder within the bag, was less than that of the gun bore. But the diameter of the gun bores was 2·02 inches; therefore,

deducting the ·05, the remainder 1·97 is the diameter of the powder cylinder within the bag: and because the areas of circles are to each other as the squares of their diameters, and the squares of these numbers, 1·97 and 2·02, being to each other as 388 to 408; or as 97 to 102; therefore, on this account alone, the numbers before found, for the value of n, must be increased in the ratio of 97 to 102.

But there is yet another circumstance, which occasions the space at first occupied by the inflamed powder to be larger than that at which it has been taken in the foregoing calculations, and that is the difference between the content of a sphere and cylinder. For, the space supposed to be occupied at first by the elastic fluid, was considered as the length of a cylinder measured to the hinder part of the curve surface of the ball, which is manifestly too little by the difference between the content of half the ball and a cylinder of the same length and diameter, that is, by a cylinder whose length is $\frac{1}{3}$ the semidiameter of the ball. Now that diameter was 1·96 inches; the half of which is 0·98, and $\frac{1}{3}$ of this is 0·33 nearly. Hence then it appears that the lengths of the cylinders, at first filled by the dense fluid, viz. 3·45, and 5·99, and 11·07, have been all taken too little by 0·33; and hence it follows that, on this account also, all the numbers before found for the value of the first force n, must be further increased in the ratios of 3·45 and 5·99 and 11·07, to the same numbers increased by 0·33, that is, to the numbers 3·78 and 6·32 and 11·40.

Compounding now these last ratios with the foregoing one, viz. 97 to 102, it produces these three, viz. the ratios of 334 and 581 and 1074, respectively to 385 and 647 and 1163. Therefore increasing the last column of numbers, for the value of n, viz. those of the 4 oz. charge in the ratio of 334 to 385, and those of the 8 oz. charge in the ratio of 581 to 647, and those of the 16 oz. charge in the ratio of 1074 to 1163, with every gun, they will be reduced to the numbers in the annexed table; where the numbers are still larger and more regular than before*.

Powder.	The Guns.			
	1	2	3	4
oz.				
4	1372	1387	1438	1430
8	1637	1677	1766	1812
16	1577	1616	1782	1784

* From the experiments of 1815, 1816, it appears that n exceeds 2000, in the best gunpowder.

Thus then at length it appears that the first force of the inflamed gunpowder, when occupying only the space at first filled with the powder, is about 1800, that is 1800 times the elasticity of the natural air, or pressure of the atmosphere, in the charges with 8 oz. and 16 oz. of powder, in the two longer guns; but somewhat less in the two shorter, probably owing to the gradual firing of gunpowder in some degree; and also less in the lowest charge 4 oz. in all the guns, which may probably be owing to the less degree of heat in the small charge. But besides the foregoing circumstances that have been noticed, or used in the calculations, there are yet several others that might and ought to be taken into the account, in order to a strict and perfect solution of the problem; such as, the counter pressure of the atmosphere, and the resistance of the air on the fore part of the ball while moving along the bore of the gun; the loss of the elastic fluid by the vent and windage of the gun; the gradual firing of the powder; the unequal density of the elastic fluid in the different parts of the space it occupies between the ball and the bottom of the bore; the difference between pressure and percussion when the ball is not laid close to the powder; and perhaps some others: on all which accounts it is probable that, instead of 1800, the first force of the elastic fluid is not less than 2000 times the strength of natural air.

Corol. From the theorem last used for the velocity of the ball and elastic fluid, viz. $v = \sqrt{(\frac{2230hd^2}{p+w}n \times \log. \frac{b}{a})} = \sqrt{\frac{8567hn}{p+w}} \times \log. \frac{b}{a}$, we may find the velocity of the elastic fluid alone, viz. by taking w, or the weight of the ball, $= 0$ in the theorem, by which it becomes barely $v = \sqrt{(\frac{8567hn}{p} \times \log. \frac{b}{a})}$, for that velocity. And by computing the several preceding examples by this theorem, supposing the value of n to be 2000, the conclusions come out a little various, being between 4000 and 5000, but most of them nearer to the latter number. So that it may be concluded that the velocity of the flame, or of the fired gunpowder, expands itself at the muzzle of the gun, at the rate of about 5000 feet per second nearly.

ON THE MOTION OF BODIES IN FLUIDS.

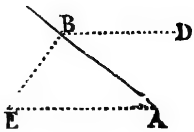
PROBLEM XIX.

To determine the Force of Fluids in Motion; and the Circumstances attending Bodies moving in Fluids.

1. It is evident that the resistance to a plane, moving perpendicularly through an infinite fluid, at rest, is equal to the pressure or force of the fluid on the plane at rest, and the fluid moving with the same velocity, and in the contrary direction, to that of the plane in the former case. But the force of the fluid in motion, must be equal to the weight or pressure which generates that motion; and which, it is known, is equal to the weight or pressure of a column of the fluid, whose base is equal to the plane, and its altitude equal to the height through which a body must fall, by the force of gravity, to acquire the velocity of the fluid: and that altitude is, for the sake of brevity, called the altitude due to the velocity. So that, if a denote the area of the plane, v the velocity, and n the specific gravity of the fluid; then, the altitude due to the velocity v being $\frac{v^2}{2g}$, the whole resistance, or motive force m, will be $a \times n \times \frac{v^2}{2g} = \frac{anv^2}{2g}$; g being, as we have all along assumed it, $= 32\frac{1}{6}$ feet. And hence, *cæteris paribus*, the resistance is as the square of the velocity.

2. This ratio, of the square of the velocity, may be otherwise derived thus. The force of the fluid in motion must be as the force of one particle multiplied by the number of them; but the force of a particle is as its velocity; and the number of them striking the plane in a given time, is also as the velocity; therefore the whole force is as $v \times v$ or v^2, that is, as the square of the velocity.

3. If the direction of motion, instead of being perpendicular to the plane, as above supposed, be inclined to it in any angle, the sine of that angle being s, to the radius 1; then the resistance to the plane, or the force of the fluid against the plane, in the direction of the motion, as assigned above, will be diminished in the triplicate ratio of radius to the sine of the angle of inclination, or in the ratio of 1 to s^3.



For, AB being the direction of the plane, and BD that of the motion, making the angle ABD, whose sine is s; the number of particles, or quantity of the fluid striking the plane, will be diminished in the ratio of 1 to s, or of radius to the sine of the angle B of inclination; and the force of each particle will also be diminished in the same ratio of 1 to s: so that, on both these accounts, the whole resistance will be diminished in the ratio of 1 to s^2, or in the duplicate ratio of radius to the sine of the said angle. But again, it is to be considered that this whole resistance is exerted in the direction BE perpendicular to the plane; and any force in the direction BE, is to its effect in the direction AE, parallel to BD, as AE to BE, that is as 1 to s. So that finally, on all these accounts, the resistance in the direction of motion, is diminished in the ratio of 1 to s^3, or in the triplicate ratio of radius to the sine of inclination. Hence, comparing this with article 1, the whose resistance, or the motive force on the plane, will be $m = \frac{anv^2s^3}{2g}$

4. Also, if w denote the weight of the body, whose plane face a is resisted by the absolute force m; then the retarding force f, or $\frac{m}{w}$, will be $\frac{anv^2s^3}{2gw}$

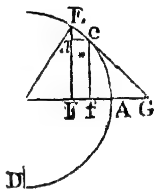
5. And if the body be a cylinder, whose face or end is a, and diameter d, or radius r, moving in the direction of its axis; because then $s = 1$, and $a = pr^2, = \frac{1}{4}pd^2$, where $p = 3 \cdot 1416$; the resisting force m will be - - - - - - $\frac{npd^2v^2}{8g} = \frac{npr^2v^2}{2g}$, and the retarding force $f = \frac{npd^2v^2}{8gw} = \frac{npr^2v^2}{2gw}$.

6. This is the value of the resistance when the end of the cylinder is a plane perpendicular to its axis, or to the direction of motion. But were its face a conical surface, or an elliptic section, or any other figure every where equally inclined to the axis, the sine of inclination being s: then the number of particles of the fluid striking the face being still the same, but the force of each opposed to the direction of motion, diminished in the duplicate ratio of radius to the sine of inclination, the resisting force m would be $\frac{npd^2v^2s^2}{8g} = \frac{npr^2v^2s^2}{2g}$.

But if the body were terminated by an end or face of any other form, as a spherical one, or such like, where every part of it has a different inclination to the axis; then a further investigation becomes necessary, such as in the following proposition.

PROBLEM XX.

To determine the Resistance of a Fluid to any Body, moving in it, of a Curved End; as a Sphere, or a Cylinder, with a Hemispherical End, &c.



1. Let BEAD be a section through the axis CA of the solid, moving in the direction of that axis. To any point of the curve draw the tangent EG, meeting the axis produced in G: also, draw the perpendicular ordinates EF, ef, indefinitely near each other; and draw ae parallel to CG.

Putting CF $= x$, EF $= y$, BE $= z$, $s =$ sine $\angle$ G to radius 1, and $p = 3\cdot1416$: then $2py$ is the circumference whose radius is EF, or the circumference described by the point E, in revolving upon the axis CA; and $2py \times$ Ee or $2py\dot{z}$ is the fluxion of the surface, or it is the surface described by Ee, in the said revolution about CA, and which is the quantity represented by a in art. 3 of the last problem: hence $\frac{nv^2s^3}{2g} \times 2py\dot{z}$ or $\frac{pnv^2s^3}{g} \times y\dot{z}$ is the resistance on that ring, or the fluxion of the resistance to the body, whatever the figure of it may be. And the fluent of which will be the resistance required.

2. In the case of a spherical form: putting the radius CA or CB $= r$, we have $y = \sqrt{(r^2 - x^2)}$, $s = \frac{\text{EF}}{\text{EG}} = \frac{\text{CF}}{\text{CE}} = \frac{x}{r}$, and $y\dot{z}$, or EF $\times$ Ee $=$ CE $\times$ ae $= r\dot{x}$; therefore the general fluxion $\frac{pnv^2}{g} \times s^3y\dot{z}$ becomes $\frac{pnv^2}{g} \times \frac{x^3}{r^3} \times r\dot{x} = \frac{pnv^2}{gr^2} \times x^3\dot{x}$; the fluent of which, or $\frac{pnv^2}{4gr^2}x^4$, is the resistance to the spherical surface generated by BE. And when x or CF is $= r$ or CA, it becomes $\frac{pnv^2r^2}{4g}$ for the resistance on the whole

hemisphere; which is also equal to $\frac{pnv^2d^2}{16g}$, where $d = 2r$ the diameter.

3. But the perpendicular resistance to the circle of the same diameter d or BD, by art. 5 of the preceding problem, is $\frac{pnv^2d^2}{8g}$; which, being double the former, shows that the resistance to the sphere, is just equal to half the direct resistance to a great circle of it, or to a cylinder of the same diameter.

4. Since $\frac{1}{6}pd^3$ is the magnitude of the globe; if N denote its density or specific gravity, its weight w will be $= \frac{1}{6}pd^3N$, and therefore the retardive force f or $\frac{m}{w} = \frac{pnv^2d^2}{16g} \times \frac{6}{pNd^3} = \frac{3nv^2}{8gNd}$; which is also $= \frac{v^2}{2gs}$ by art. 8 of the general theorems in page 370; hence then $\frac{3n}{4Nd} = \frac{1}{s}$, and $s = \frac{N}{n} \times \frac{4}{3}d$; which is the space that would be described by the globe, while its whole motion is generated or destroyed by a constant force which is equal to the force of resistance, if no other force acted on the globe to continue its motion. And if the density of the fluid were equal to that of the globe, the resisting force is such, as, acting constantly on the globe without any other force, would generate or destroy its motion in describing the space $\frac{4}{3}d$, or $\frac{4}{3}$ of its diameter, by that accelerating or retarding force.

5. Hence the greatest velocity that a globe will acquire by descending in a fluid, by means of its relative weight in the fluid, will be found by making the resisting force equal to that weight. For, after the velocity is arrived at such a degree, that the resisting force is equal to the weight that urges it, it can increase no longer, and the globe will afterwards continue to descend with that velocity uniformly. Now, N and n being the separate specific gravities of the globe and fluid, N $- n$ will be the relative gravity of the globe in the fluid, and therefore $w = \frac{1}{6}pd^3$ (N $- n$) is the weight by which it is urged; also $m =$ - - - - - $\frac{pnv^2d^2}{16g}$ is the resistance; consequently $\frac{pnv^2d^2}{16g} = \frac{1}{6}pd^3$ (N $- n$) when the velocity becomes uniform; from which equation

is found $v = \sqrt{(2g \cdot \frac{4}{3}d \cdot \frac{N-n}{n})}$, for the said uniform or greatest velocity.

And, by comparing this form with that in art. 6 of the general theorems in page 370, it will appear that its greatest velocity is equal to the velocity generated by the accelerating force $\frac{N-n}{n}$, in describing the space $\frac{4}{3}d$, or equal to the velocity generated by gravity in freely describing the space $\frac{N-n}{n} \times \frac{4}{3}d$. If $N = 2n$, or the specific gravity of the globe be double that of the fluid, then $\frac{N-n}{n} = 1 =$ the natural force of gravity; and then the globe will attain its greatest velocity in describing $\frac{4}{3}d$ or $\frac{4}{3}$ of its diameter.—It is further evident, that if the body be very small, it will very soon acquire its greatest velocity, whatever its density may be.

Exam. If a leaden ball, of 1 inch diameter, descend in water, and in air of the same density as at the earth's surface, the three specific gravities being as $11\frac{1}{3}$, and 1, and $\frac{3}{2500}$. Then $v = \sqrt{(4 \cdot 16\frac{1}{12} \cdot \frac{4}{36} \cdot 10\frac{1}{3})} = \frac{1}{9}\sqrt{(31 \cdot 193)} = 8{\cdot}5944$ feet, is the greatest velocity per second the ball can acquire by descending in water. And $v = \sqrt{(4 \cdot \frac{193}{12} \cdot \frac{4}{36} \cdot \frac{34}{3} \cdot \frac{2500}{3})}$ nearly $= \frac{50}{9}\sqrt{\frac{34 \cdot 193}{3}} = 259{\cdot}82$ is the greatest velocity it can acquire in air.

But if the globe were only $\frac{1}{100}$ of an inch diameter, the greatest velocities it could acquire, would be only $\frac{1}{10}$ of these, namely $\frac{86}{100}$ of a foot in water, and 26 feet nearly in air. And if the ball were still further diminished, the greatest velocity would also be diminished, and that in the subduplicate ratio of the diameter of the ball.

PROBLEM XXI.

To determine the Relations of Velocity, Space, and Time, of a Ball moving in a Fluid, in which it is projected with a given Velocity.

1. Let $a =$ the first velocity of projection, x the space described in any time t, and v the velocity then. Now, by art. 4 of the last problem, the accelerative force $f = \frac{3nv^2}{8gNd}$,

where n is the density of the fluid, N that of the ball, and d its diameter. Therefore the general equation $v\dot{v} = gf\dot{s}$ becomes $v\dot{v} =$ - - - - - - - - -
$\frac{-3nv^2}{8Nd}\dot{x}$; and hence $\frac{\dot{v}}{v} = \frac{-3n}{8Nd}\dot{x} = -b\dot{x}$, putting b for $\frac{3n}{8Nd}$.

The correct fluent of this, is log. a − log. v or log. $\frac{a}{v} = bx$.

Or, putting $c = 2{\cdot}718281828$, the number whose hyp. log. is 1, then is $\frac{a}{v} = c^{bx}$, and the velocity $v = \frac{a}{c^{bx}} = ac^{-bx}$.

2. The velocity v at any time being the c^{-bx} part of the first velocity, therefore the velocity lost in any time, will be the $1 - c^{-bx}$ part, or the $\frac{c^{bx}-1}{c^{bx}}$ part of the first velocity.

EXAMPLES.

EXAM. 1. If a globe be projected, with any velocity, in a medium of the same density with itself, and it describe a space equal to $3d$ or 3 of its diameters. Then $x = 3d$, and $b = \frac{3n}{8Nd} = \frac{3}{8d}$; therefore $bx = \frac{9}{8}$, and $\frac{c^{bx}-1}{c^{bx}} = \frac{2{\cdot}08}{3{\cdot}08}$ is the velocity lost, or nearly $\frac{2}{3}$ of the projectile velocity.

EXAM. 2. If an iron ball of 2 inches diameter were projected with a velocity of 1200 feet per second; to find the velocity lost after moving through any space, as suppose 500 feet of air: we should have $d = \frac{2}{12} = \frac{1}{6}$, $a = 1200$, $x = 500$, $N = 7\frac{1}{3}$, $n = {\cdot}0012$; and therefore $bx =$ - -
$\frac{3nx}{8Nd} = \frac{3\ .\ 12\ .\ 500\ .\ 3\ .\ 6}{8\ .\ 22\ .\ 10000} = \frac{81}{440}$, and $v = \frac{1200}{c^{\frac{81}{440}}} = 998$ feet per second: having lost 202 feet, or nearly $\frac{1}{6}$ of its first velocity.

EXAM. 3. If the earth revolved about the sun, in a medium as dense as the atmosphere near the earth's surface; and it were required to find the quantity of motion lost in a year. Then, if the earth's mean density be about $4\frac{1}{2}$, and its distance from the sun 12000 of its diameters, we have $24000 \times 3{\cdot}1416 = 75398$ diameters $= x$, and $bx =$ - -
$\frac{3\ .\ 75398\ .\ 12\ .\ 2}{8\ .\ 10000\ .\ 9} = 7{\cdot}5398$; hence $\frac{c^{bx}-1}{c^{bx}} = \frac{1880}{1881}$ parts

are lost of the first motion in the space of a year, and only the $\frac{1}{1881}$ part remains. If the earth's mean density be taken $= 5$, the result will become $\frac{885}{886}$ for the motion lost.

EXAM. 4. If it be required to determine the distance moved, x, when the globe has lost any part of its motion, as suppose $\frac{1}{2}$, and the density of the globe and fluid equal; the general equation gives $x = \frac{1}{b} \times \log. \frac{a}{v} = \frac{8d}{3} \times \log.$ of $2 = 1{\cdot}84839925d$. So that the globe loses half its motion before it has described twice its diameter.

3. To find the time t; we have $\dot{t} = \frac{\dot{s}}{v} = \frac{\dot{x}}{v} = \frac{c^{bx}\dot{x}}{a}$. Now, to find the fluent of this, put $z = c^{bx}$; then is $bx =$ log. z, and $b\dot{x} = \frac{\dot{z}}{z}$, or $\dot{x} = \frac{\dot{z}}{bz}$; conseq. $\dot{t}$ or $\frac{c^{bx}\dot{x}}{a} = \frac{z\dot{x}}{a} = \frac{\dot{z}}{ab}$; and hence $t = \frac{z}{ab} = \frac{c^{bx}}{ab}$. But as t and x vanish together, and when $x = 0$, the quantity $\frac{c^{bx}}{ab}$ is $= \frac{1}{ab}$; therefore, by correction, $t = \frac{c^{bx} - 1}{ab} = \frac{1}{bv} - \frac{1}{ba} = \frac{1}{b}(\frac{1}{v} - \frac{1}{a})$ the time sought; where $b = \frac{3n}{8Nd}$, and $v = \frac{a}{c^{bx}}$ the velocity.

EXAM. If an iron ball of 2 inches diameter were projected in the air with a velocity of 1200 feet per second; and it were required to determine in what time it would pass over 500 yards or 1500 feet, and what would be its velocity at the end of that time? We should have, as in exam. 2 above, $b = \frac{3 \cdot 12 \cdot 3 \cdot 6}{8 \cdot 22 \cdot 10000} = \frac{1}{2716}$, and $bx = \frac{1500}{2716} = \frac{375}{679}$; hence $\frac{1}{b} = \frac{2716}{1}$, and $\frac{1}{a} = \frac{1}{1200}$, and $\frac{1}{v} = \frac{c^{bx}}{a} = \frac{1{\cdot}7372}{1200} = \frac{1}{690}$ nearly. Consequently $v = 690$ is the velocity; and $t = \frac{1}{b}(\frac{1}{v} - \frac{1}{a}) = 2716 \times (\frac{1}{690} - \frac{1}{1200}) = 1\frac{31}{46}$ seconds, is the time required, or $1''$ and $\frac{2}{3}$ nearly.

PROBLEM XXII.

To determine the Relations of Space, Time, and Velocity, when a Globe descends, by its own Weight, in a Fluid.

The foregoing notation remaining, viz. $d =$ diameter,

N and n the density of the ball and fluid, and v, s, t, the velocity, space, and time, in motion; we have $\frac{1}{6}pd^3$ = the magnitude of the ball, and $\frac{1}{6}pd^3(N - n)$ = its weight in the fluid, also $m = \frac{pnd^2v^2}{16g}$ = its resistance from the fluid; consequently $\frac{1}{6}pd^3(N - n) - \frac{pnd^2v^2}{16g}$ is the motive force by which the ball is urged; which being divided by $\frac{1}{6}pNd^3$, the quantity of matter moved, gives $f = 1 - \frac{n}{N} - \frac{3nv^2}{8gNd}$ for the accelerative force.

2. Hence $v\dot{v} = gf\dot{s}$, and $\dot{s} = \frac{v\dot{v}}{gf} = \frac{Nv\dot{v}}{g(N - n) - \frac{3n}{8d}v^2}$ $= \frac{1}{b} \times \frac{v\dot{v}}{a - v^2}$, putting $b = \frac{3n}{8Nd}$, and $\frac{1}{a} = \frac{3N}{g\,.\,8d(N - n)}$, or $ab = g$ nearly; the fluent of which is s = - - - - - $\frac{1}{2b} \times$ log. of $\frac{a}{a - v^2}$, an expression for the space s, in terms of the velocity v. That is, when s and v begin, or are equal to nothing, both together.

But if the body commence motion in the fluid with a certain given velocity c, or enter the fluid with that velocity, like as when the body, after falling in empty space from a certain height, falls into a fluid like water; then the correct fluent will be $s = \frac{1}{2b} \times$ hyp. log. of $\frac{a - c^2}{a - v^2}$.

3. But now, to determine v in terms of s, put c = 2·718281828; then, since the log. of $\frac{a}{a - v^2} = 2bs$, therefore $\frac{a}{a - v^2} = c^{2bs}$, or $\frac{a - v^2}{a} = c^{-2bs}$; hence $v = \sqrt{a - ac^{-2bs}}$ is the velocity sought.

4. The greatest velocity is to be found, as in art. 5 of prob. 20, by making f or $1 - \frac{n}{N} - \frac{3nv^2}{8gNd} = 0$, which gives $v = \sqrt{g\,.\,8d\,.\,\frac{N - n}{3n}} = \sqrt{a}$. The same value of v is obtained by making the fluxion of v^2, or of $a - ac^{-2bs}$, $= 0$.

And the same value of v is also obtained by making s infinite, for then $c^{-2bs} = 0$. But this velocity $\sqrt{a}$ cannot be attained in any finite time, and it only denotes the velocity to which the general value of v or $\sqrt{a - ac^{-2bs}}$ continually approaches. It is evident however, that it will approximate towards it the faster, the greater b is, or the less d is; and that, the diameters being very small, the bodies descend by nearly uniform velocities, which are directly in the subduplicate ratio of the diameters. See also art. 5, prob. 20, for other observations on this head.

5. To find the time t. Now $\dot{t} = \frac{\dot{s}}{v} = \sqrt{\frac{1}{a}} \times \frac{\dot{s}}{\sqrt{(1-c^{-2bs})}}$

Then, to find the fluent of this fluxion, put $z = \sqrt{(1-c^{-2bs})}$ $= \frac{v}{\sqrt{a}}$, or $z^2 = 1 - c^{-2bs}$; hence $z\dot{z} = b\dot{s}c^{-2bs}$, and $\dot{s} = \frac{z\dot{z}}{bc^{-2bs}}$ $= \frac{1}{b} \cdot \frac{z\dot{z}}{1-z^2}$, consequently $\dot{t} = \frac{1}{b\sqrt{a}} \cdot \frac{\dot{z}}{1-z^2}$, - - -

and therefore the fluent is $t = \frac{1}{2b\sqrt{a}} \times \log. \frac{1+z}{1-z} = \frac{1}{2b\sqrt{a}}$ $\times \log. \frac{1+\sqrt{(1-c^{-2bs})}}{1-\sqrt{(1-c^{-2bs})}} = \frac{1}{2b\sqrt{a}} \times \log. \frac{\sqrt{a}+v}{\sqrt{a}-v}$, which is the general expression for the time.

Or thus: because $\dot{s} = \frac{1}{b} \cdot \frac{v\dot{v}}{a-v^2}$, theref. $\dot{t} = \frac{1}{b} \cdot \frac{\dot{v}}{a-v^2}$, and the fluent, by form 10, is $\frac{1}{2b\sqrt{a}} \times \log. \frac{\sqrt{a}+v}{\sqrt{a}-v}$.

Exam. If it were required to determine the time and velocity, by descending in air 1000 feet, the ball being of lead, and 1 inch diameter.

Here $N = 11\frac{1}{3}$, $n = \frac{3}{2500}$, $d = \frac{1}{12}$, and $s = 1000$.

Hence $a = \frac{2 \cdot 16 \cdot \frac{1}{12} \cdot \frac{3}{16} \cdot 11\frac{1}{3}}{3 \cdot \frac{3}{2500}} = \frac{2 \cdot 193 \cdot 8 \cdot 34 \cdot 2500}{3 \cdot 3 \cdot 12 \cdot 12 \cdot 3} =$ $\frac{193 \cdot 34 \cdot 50^2}{9 \cdot 27}$, and $b = \frac{3 \cdot \frac{3}{2500}}{8 \cdot 11\frac{1}{3} \cdot \frac{1}{12}} = \frac{3 \cdot 3 \cdot 3 \cdot 12}{8 \cdot 34 \cdot 2500} = \frac{9 \cdot 9}{68 \cdot 50^2}$;

consequently $v = \sqrt{a} \times \sqrt{(1 - c^{-2bs})} = \sqrt{\frac{193 \cdot 34 \cdot 50^2}{9 \cdot 27}} \times$ $\sqrt{(1 - c^{-\frac{81}{85}})} = 203\frac{2}{3}$ the velocity. And $t = \frac{1}{2b\sqrt{a}} \times \log.$

$$\frac{2+\sqrt{1-c^{-2bs}}}{1-\sqrt{1-c^{-2bs}}} = \sqrt{\frac{34\,.\,2500}{27\,.\,193}} \times \log.\frac{1\cdot78383}{0\cdot21617} = 8\cdot5236'',$$

the time.

Note. If the globe be so light as to ascend in the fluid; it is only necessary to change the signs of the first two terms in the value of f, or the accelerating force, by which it becomes $f = \frac{n}{\text{N}} - 1 - \frac{3nv^2}{8g\text{N}d}$; and then proceed in all respects as before.

SCHOLIUM.

To compare this theory, contained in the last four problems, with experiment, the few following numbers are here extracted from extensive tables of velocities and resistances, resulting from a course of many hundred very accurate experiments, made in the course of the year 1786.

In the first column are contained the mean uniform or greatest velocities acquired in air, by globes, hemispheres, cylinders, and cones, all of the same diameter, and the altitude of the cone nearly equal to the diameter also, when urged by the several weights expressed in avoirdupois ounces, and standing on the same line with the velocities, each in their proper column. So, in the first line, the numbers show, that, when the greatest or uniform velocity was accurately 3 feet per second, the bodies were urged by these weights, according as their different ends went foremost; namely, by ·028 oz. when the vertex of the cone went foremost; by ·064 oz. when the base of the cone went foremost; by ·027 oz. for a whole sphere; by ·050 oz. for a cylinder; by ·051 oz. for the flat side of the hemisphere; and by ·020 oz. for the round or convex side of the hemisphere. Also, at the bottom of all, are placed the mean proportions of the resistances of these figures in the nearest whole numbers. Note, the common diameter of all the figures was 6·375, or $6\frac{3}{8}$ inches; so that the area of the circle of that diameter is just 32 square inches, or $\frac{2}{9}$ of a square foot; and the altitude of the cone was $6\frac{5}{8}$ inches. Also, the diameter of the small hemisphere was $4\frac{3}{4}$ inches, and consequently the area of its base $17\frac{3}{4}$ square inches, or $\frac{1}{8}$ of a square foot nearly.

From the given dimensions of the cone, it appears, that the angle made by its side and axis, or direction of the path, is 25° 42′, very nearly.

The mean height of the barometer at the times of making the experiments, was nearly 30·1 inches, and of the thermometer 62°; consequently the weight of a cubic foot of air was equal to $1\frac{1}{5}$ oz. nearly, in those circumstances.

Veloc. per sec.	Cone. vertex.	Cone. base.	Whole globe.	Cylinder.	Hemisphere. flat.	Hemisphere. round.	Small Hemis. flat.
feet.	oz.	oz.	oz.	oz.	oz.	oz.	oz.
3	·028	·064	·027	·050	·051	·020	·028
4	·048	·109	·047	·090	·096	·039	·048
5	·071	·162	·068	·143	·148	·063	·072
6	·098	·225	·094	·205	·211	·092	·103
7	·129	·298	·125	·278	·284	·123	·141
8	·168	·382	·162	·360	·368	·160	·184
9	·211	·478	·205	·456	·464	·199	·233
10	·260	·587	·255	·565	·573	·242	·287
11	·315	·712	·310	·688	·698	·297	·349
12	·376	·850	·370	·826	·836	·347	·418
13	·440	1·000	·435	·979	·988	·409	·492
14	·512	1·166	·505	1·145	1·154	·478	·573
15	·589	1·346	·581	1·327	1·336	·552	·661
16	·673	1·546	·663	1·526	1·538	·634	·754
17	·762	1·763	·752	1·745	1·757	·722	·853
18	·858	2·002	·848	1·986	1·998	·818	·959
19	·959	2·260	·949	2·246	2·258	·922	1·073
20	1·069	2·540	1·057	2·528	2·542	1·033	1·196
Propor. Numb.	126	291	124	285	288	119	140

From this table of resistances, several practical inferences may be drawn. As,

1. That the resistance is nearly as the surface; the resistance increasing but a very little above that proportion in the greater surfaces. Thus, by comparing together the numbers in the 6th and last columns, for the bases of the two hemispheres, the areas of which are in the proportion of $17\frac{3}{4}$ to 32, or as 5 to 9 very nearly; it appears that the numbers in those two columns, expressing the resistances, are nearly as 1 to 2, or as 5 to 10, as far as to the velocity of 12 feet; after which the resistances on the greater surface increase gradually more and more above that proportion. And the

mean resistances are as 140 to 288, or as 5 to $10\frac{2}{7}$. This circumstance therefore agrees nearly with the theory.

2. The resistance to the same surface, is nearly as the square of the velocity; but gradually increasing more and more above that proportion, as the velocity increases. This is manifest from all the columns. And therefore this circumstance also differs but little from the theory, in small velocities.

3. When the hinder parts of bodies are of different forms, the resistances are different, though the fore parts be alike; owing to the different pressures of the air on the hinder parts. Thus, the resistance to the fore part of the cylinder, is less than that on the flat base of the hemisphere, or of the cone; because the hinder part of the cylinder is more pressed or pushed, by the following air, than those of the other two figures.

4. The resistance on the base of the hemisphere, is to that on the convex side, nearly as $2\frac{2}{5}$ to 1, instead of 2 to 1, as the theory assigns the proportion. And the experimented resistance, in each of these, is nearly $\frac{1}{4}$ part more than that which is assigned by the theory.

5. The resistance on the base of the cone is to that on the vertex, nearly as $2\frac{3}{10}$ to 1. And in the same ratio is radius to the sine of the angle of the inclination of the side of the cone, to its path or axis. So that, in this instance, the resistance is directly as the sine of the angle of incidence, the transverse section being the same, instead of the square of the sine.

6. Hence we can find the altitude of a column of air, whose pressure shall be equal to the resistance of a body, moving through it with any velocity. Thus,

Let $a =$ the area of the section of the body, similar to any of those in the table, perpendicular to the direction of motion;
$r =$ the resistance to the velocity, in the table; and
$x =$ the altitude sought, of a column of air, whose base is a, and its pressure r.

Then $ax =$ the content of the column in feet,
and $1\frac{1}{5}ax$ or $\frac{6}{5}ax$ its weight in ounces; - - - -

therefore $\frac{6}{5}ax = r$, and $x = \frac{5}{6} \times \frac{r}{a}$ is the altitude sought in feet, namely, $\frac{5}{6}$ of the quotient of the resistance of any body divided by its transverse section; which is a constant quantity for all similar bodies, however different in magnitude,

since the resistance r is as the section a, as was found in art. 1. When $a = \frac{2}{9}$ of a foot, as in all the figures in the foregoing table, except the small hemisphere: then, $x = \frac{5}{6} \times \frac{r}{a}$ becomes $x = \frac{15}{4}r$, where r is the resistance in the table, to the similar body.

If, for example, we take the convex side of the large hemisphere, whose resistance is ·634 oz. to a velocity of 16 feet per second, then $r = ·634$, and $x = \frac{15}{4}r = 2·3775$ feet, is the altitude of the column of air whose pressure is equal to the resistance on a spherical surface, with a velocity of 16 feet. And to compare the above altitude with that which is due to the given velocity, it will be $32^2 : 16^2 :: 16 : 4$, the altitude due to the velocity 16; which is near double the altitude that is equal to the pressure. And as the altitude is proportional to the square of the velocity, therefore, in small velocities, the resistance to any spherical surface, is equal to the pressure of a column of air on its great circle, whose altitude is $\frac{19}{32}$ or ·594 of the altitude due to its velocity.

But if the cylinder be taken, whose resistance $r = 1·526$: then $x = \frac{15}{4}r = 5·72$; which exceeds the height, 4, due to the velocity in the ratio of 23 to 16 nearly. And the difference would be still greater, if the body were larger; and also if the velocity were more.

7. Also, if it be required to find with what velocity any flat surface must be moved, so as to suffer a resistance just equal to the whole pressure of the atmosphere:

The resistance on the whole circle whose area is $\frac{2}{9}$ of a foot, is ·051 oz. with the velocity of 3 feet per second; it is $\frac{1}{9}$ of ·051, or ·0056 oz. only, with a velocity of 1 foot. But $2\frac{1}{2} \times 13600 \times \frac{2}{9} = 7555\frac{5}{9}$ oz. is the whole pressure of the atmosphere. Therefore, as $\sqrt{·0056} : \sqrt{7556} :: 1 : 1162$ nearly, which is the velocity sought. Being almost equal to the velocity with which air rushes into a vacuum.

8. Hence may be inferred the great resistance suffered by military projectiles. For, in the table, it appears, that a globe of $6\frac{3}{8}$ inches diameter, which is equal to the size of an iron ball weighing 36lb. moving with a velocity of only 16 feet per second, meets with a resistance equal to the pressure of $\frac{2}{3}$ of an ounce weight; and therefore, computing only according to the square of the velocity, the least resistance that such a ball would meet with, when moving with a velocity of 1600 feet, would be equal to the pressure of 417lb., and that independent of the pressure of the atmo-

sphere itself on the fore part of the ball, which would be 487lb. more, as there would be no pressure from the atmosphere on the hinder part, in the case of so great a velocity as 1600 feet per second. So that the whole resistance would be more than 900lb. to such a velocity.

9. Having said, in the last article, that the pressure of the atmosphere is taken entirely off the hinder part of the ball moving with a velocity of 1600 feet per second; which must happen when the ball moves faster than the particles of air can follow by rushing into the place quitted and left void by the ball, or when the ball moves faster than the air rushes into a vacuum from the pressure of the incumbent air: let us therefore inquire what this velocity is. Now the velocity with which any fluid issues, depends on its altitude above the orifice, and is indeed equal to the velocity acquired by a heavy body in falling freely through that altitude. But, supposing the height of the barometer to be 30 inches, or $2\frac{1}{2}$ feet, the height of a uniform atmosphere, all of the same density as at the earth's surface, would be $2\frac{1}{2} \times 14 \times 833\frac{1}{3}$ or 29167 feet; therefore $\sqrt{16} : \sqrt{29167} :: 32 : 8\sqrt{29167} = 1366$ feet, which is the velocity sought. And therefore, with a velocity of 1600 feet per second, or any velocity above 1366 feet, the ball must continually leave a vacuum behind it, and so must sustain the whole pressure of the atmosphere on its fore part, as well as the resistance arising from the *vis inertia* of the particles of air struck by the ball.

10. On the whole, we find that the resistance of the air, as determined by the experiments, differs very widely, both in respect to its quantity on all figures, and in respect to the proportions of it on oblique surfaces, from the same as determined by the preceding theory; which accords with that of Sir Isaac Newton, and most modern philosophers. Neither should we succeed better if we have recourse to the theory given by Professor Gravesande, or others, as similar differences and inconsistencies still occur.

We conclude therefore, that all the theories of the resistance of the air hitherto given, are very erroneous. And the preceding one is only laid down, till further experiments, on this important subject, shall enable philosophers to deduce from them another, that shall be more consonant to the true phænomena of nature.

APPENDIX.

TABLES, &c. OF COMPARATIVE STRENGTH;

OR, THE SPECIFIC COHESION OF DIFFERENT SUBSTANCES.

(From Tables drawn up by Mr. Thomas Tredgold.)

It is the cohesion of the parts of solid bodies, which not only serves to characterise different substances, but also to determine their relative value in the various uses to which they may be appropriated. The standard degree of cohesion, employed in the following tables, is plate-glass, which is taken as *unity*, and the other substances are stronger or weaker in proportion as they are above or below 1. The strength of woods of the same kinds is, it will be observed, extremely variable, depending on the age, the nature of the soil, and the situation and the climate where they are grown.

TABLE I.—*Woods.*

	Specific Cohesion.		Specific Cohesion.
Lance-wood -	2·621	Oak - -	1·836
Locust-tree - -	2·185	Dry, cut 4 years -	1·707
Jujube (Ziziphus)	2·008	Provence, seasoned*	1·559
ASH (Fraxinus).		English, seasoned	1·509
Red, seasoned -	1·899	Oak - -	1·481
Ash - - -	1·804	French, seasoned †	1·450
White, seasoned -	1·509	Provence, seasoned ‡	1·444
Ash - -	1·274	Provence, seasoned, young - -	1·363
OAK (Quercus) -	1·891	Oak, dry - -	1·274
——, highest result	1·861		

* Its colour brown, and it was hard and large-veined.

† This specimen lay six months in water after it was cut, and was afterwards dried. When the trial was made, it had been cut four years.

‡ Middle-aged timber, fine-veined, light and pliant.

	Specific Cohesion.
Baltic, seasoned	1·211
Oak, lowest result	1·146
——,	1·107
English	1·085
Oak	1·076
French, unseasoned	1·060
White American, seasoned	1·009
Oak	1·009
French, unseasoned	0·960
Oak	0·955
English	0·936
Dantzic	0·818
Beech (Fagus sylvaticus)	1·880
Arbutus, from	1·845
to	0·814
Orange (Aurantium)	1·764
to	1·629
Bay (Laurus)	1·547
to	1·085
TEAK (Tectona grandis.)	
Java, seasoned	1·509
Pegu, seasoned	1·400
Malabar, seasoned	1·395
Alder (Bet. Alnus)	1·506
Mulberry (Morus)	1·492
to	1·221
Elm (Ulmus)	1·432
FIRS (Pinus.)	
Pitch pine	1·398
Fir	1·380
Fir (strongest)	1·318
Pitch pine	1·284
Pine (Pin du Nord)	1·264
Larch (Pinus Larix)	1·177
Fir, strong red	1·172
Fir, Memel, seasoned	1·154
Fir, Russian	1·062
Fir	1·061
Fir	1·039
Fir, Riga	0·963
Fir, American	0·942
Fir	0·903

	Specific Cohesion.
Fir, yellow deal	0·900
Fir, weakest	0·879
Larch, Scotch, seasoned	0 837
Pitch pine	0·830
Larch, Scotch, very dry	0·745
Fir, Scotch (P. sylvestris)	0·711
Fir, white deal	0·455
Sissor, of Bengal	1·395
Saul, of Bengal	1·375
Plum, (Prunus)	1·357
to	1·205
Willow, (Salix)	1·357
Willow, dry	0·809
MAHOGANY (Swietenia).	
Spanish	1·283
Citron (Citreum)	1·357
to	0·868
CHESTNUT, Sweet (Fagus castanea.)	
100 years in use	1·291
Jasmine (Jasminum)	1·276
to	1·248
Pomegranate (Punica)	1·221
to	0·882
Tamarisk (Tamariscus)	1·194
to	0·732
MAPLE (Acer.)	
Norway	1·123
Elder (Sambucus)	1·086
Lemon (Limon)	1·004
Quince (Cydonia)	0·841
to	0·624
Cypress (Cupressus)	0·732
to	0·542
Poplar (Pop. alba)	0·705
to	0·488
Poplar (P. nigra) lateral cohesion of the animal rings	0·189
Cedar	0·528

TABLE II.—*Comparative Strength of Metals.*

(*h*) and (*l*) mark the highest and lowest result obtained from each kind of iron.

	Specific Cohesion. Pl. Glass as 1.
STEEL.	
Razor temper	15·927
Soft	12·739
IRON.	
Wire	12·004
German bar, mark BR (*h*)	9·880
Swedish bar (*h*)	9·445
German bar, mark L (*h*)	9·119
Wire	9·108
Bar	8·964
Liege bar (*h*)	8·794
Spanish bar	8·685
Bar	8·581
Bar	8·492
Oosement bar (*h*)	8·142
Cable	7·752
German bar, mark L (*l*)	7·382
German bar, common	7·339
Swedish bar, Oosement bar (*l*)	7·296
Bar of best quality	7·006
Liege bar (*l*)	6·621
German bar, mark BR (*l*)	6·514
Bar*	6·480
Bar of good quality	5·839
Cable	5·787
Bar, fine-grained	5·306
——, medium fineness	3·618
——, coarse-grained	2·172

	Specific Cohesion. Pl. Glass as 1.
CAST IRON.	
French	7·470
German	7·250
French, soft	6·754
English	5·520
French	5·412
——	4·540
English, soft	4·334
French gray	4·000
Gray, of Cruzot, 2nd fusion	3·257
Gray, of Cruzot, 1st fusion	3·202
COPPER.	
Wire	6·606
Cast, Barbary	2·396
——, Japan	2·152
PLATINUM.	
Wire	5·995
Wire	5·625
SILVER.	
Wire	4·090
Cast	4·342
GOLD.	
Wire	3·279
Cast	2·171
TIN.	
Wire	0·7568
Cast, English block	0·706

* This is the mean result of thirty-three experiments.

	Specific Cohesion. Pl. Glass as 1.
Cast, English block	0·565
——, Banca -	0·3906
——, Malacca -	0·342
Bismuth.	
Cast - -	0·345
—— - -	0·3193
Zinc.	
Wire - -	2·394
Patent sheet -	1·762

	Specific Cohesion. Pl. Glass as 1.
Cast, Goslar, from	0·3118
to -	0·2855
Lead.	
Milled - -	0·3538
Wire - -	0·334
Wire - -	0·274
Wire - -	0·2704
Cast, English -	0·094
Antimony, cast -	0·1126

Table III.—*Comparative Strength of Marble, Ivory, and other Miscellaneous Substances.*

	Specific Cohesion. Glass as 1.
Hemp fibres glued together -	9·766
Paper strips glued together -	3·184
Ivory - -	1·765
Slate, Welsh, (clay slate) - -	1·358
Plate-glass -	1·000
Marble (white) -	0·955
Horn of an ox -	0·950
Whalebone -	0·814
Bone of an ox -	0·559
Hard stone* of Givry - -	0·230
Portland stone (compact lime-stone)	0·083
Soft stone† of Givry - -	0·041
Brick from -	0·031
to - -	0·030
Brick from Dorking	0·029
Stone, homogeneous white, of a fine grain -	0·022
Plaster of Paris -	0·0077
Mortar of sand and lime, 16 years made - -	0·0054

Comparative Strength of Substances.

Note. If any of the numbers in these tables be multiplied by 9420, the product will express the force in pounds avoirdupois that would tear asunder a bar of the respective substance *an inch square.* By this process, therefore, these

* This stone was hard, of a red colour, and the beds distinctly marked.

† This stone was white, rather soft, and the beds not distinctly marked. These numbers were calculated from experiments on the transverse strength.

numbers will furnish values of c, similar in nature, and applicable to the same purposes, as the values of c, in the table at pa. 362.

Thus, $6{\cdot}754 \times 9420 = 63622{\cdot}68$, value of c for French soft cast iron.

Again, $1{\cdot}123 \times 9420 = 10578{\cdot}66$, value of c for Norway maple.

And $\cdot 3538 \times 9420 = 3332{\cdot}796$, value of c for milled lead. And so with regard to others.

*Practical Rules for ascertaining the Dimensions of Gudgeons, Shafts, &c.**

1. Let w utmost amount of the stress in cwts., l the length of the gudgeon in inches from the shoulder to the extreme point of bearing, d the diameter of the gudgeon in inches; then

$$0{\cdot}42\,(\mathrm{w}l)^{\frac{1}{3}} = d.$$

2. If a cylindrical shaft have no other lateral stress to sustain than its own weight, then the rule is $\sqrt{(\cdot 007 l^3)} = d$, diameter of the shaft in inches.

3. Let the stress, supposed to be at the middle, be n times the weight of the shaft; then

$$\sqrt{(\cdot 012 l^3 n)} = d, \text{ in inches.}$$

4. For *hollow* cylindrical shafts of cast iron, to resist lateral stress, let D, the exterior diameter, and ND, the interior one; then w being, as before, in cwts.

$$\left[\frac{\mathrm{w}l^2}{2(1-\mathrm{N}^4)}\right]^{\frac{1}{4}} = \mathrm{D}, \text{ diameter in inches.}$$

5. If the hollow shaft support n times its own weight; then

$$\sqrt{\frac{\cdot 012 l^3 n}{1+\mathrm{N}^2}} = \mathrm{D}.$$

6. For *wrought iron* shafts, find the adequate diameter for *cast iron*, and multiply by $\cdot 935$.

7. For *oak* shafts, multiply the adequate dimension for *cast* iron by $1{\cdot}83$.

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* These are selected, for their obvious utility, from Tredgold's additions to Buchanan's Essays on Millwork, &c. See farther the rules and examples at pa. 345. vol. iii. of this Course.

8. For *fir* shafts, multiply the requisite dimension for *cast* iron by 1·716.

9. For cylindrical shafts of cast iron to resist torsion, let H be the number of horses' power, N the revolutions of the shaft in a minute; then,

$$\sqrt[3]{\frac{240\text{H}}{\text{N}}} = d, \text{ in inches.}$$

10. For *wrought* iron multiply the preceding result by 0·963.

11. For *oak*, the multiplier is 2·238.

12. For *fir*, the multiplier is 2·06.

13. If a shaft have to sustain both lateral stress and torsion, the sum of the straining forces must be taken. The practical rule to be then employed, for *cast* iron, is

$$\left(\frac{240\text{H}}{\text{N}} + \frac{\text{w}l^2}{2}\right)^{\frac{1}{3}} = d, \text{ in inches.}$$

EXAMPLES.

1. Let the length of a cast iron shaft be 12 feet, the lateral stress double its own weight. Required the diameter.

Ans. 6 44 inches.

2. Supposing the length the same, and the lateral stress quadruple its own weight. Required the diameter.

Ans. 9·1 inches.

3. Required the corresponding diameters of shafts of *oak*, and of *fir*, in both cases.

Ans. in the first, for *oak* 11·78, for *fir* 11·05.
In the second, for *oak* 16·65, for *fir* 15·62.

4. Let the interior diameter of a hollow cast iron shaft, of 12 feet long, be six-tenths of the exterior diameter, and the stress four times the weight of the shaft: required both diameters.

Ans. exterior 7·9, interior 4·7 inches.

5. Let the moving force be equal to 7 horses, the number of turns per minute $11\frac{1}{2}$: required the diameter of the shaft to resist the torsion, both for cast iron and fir.

Ans. cast iron 5·267, fir 10·850 inches.

6. Suppose that a cylindrical shaft of cast iron is to make 34 revolutions per minute, the power of the first mover being

equal to three horses, the length of the shaft 8 feet, and the lateral stress 3 cwts. when reduced to the middle point. Required the diameter of the shaft.

Ans. $\left(\frac{240 \times 3}{34} + \frac{3 \cdot 8^2}{2}\right)^{\frac{1}{3}} = \sqrt[3]{(21 \cdot 18 + 96)} = 4 \cdot 893$ inches.

ON MODELS.

From an experiment made to ascertain the firmness of the model of a machine, or of an edifice, certain precautions are necessary before we can infer the firmness of the structure itself.

The classes of forces must be distinguished; as, whether they tend to *draw* asunder the parts, to *break* them transversely, or to *crush* them by compression. To the first class belongs the stretching suffered by key-stones, or bonds of vaults, &c.; to the second, the load which tends to bend or break horizontal or inclined beams; to the third, the weight which presses vertically upon walls and columns.

Prop. 1. If the side of a model be to the corresponding side of the structure as 1 to n, the stress which tends to *draw* asunder, or to *break transversely*, the parts, increases from the smaller to the greater scale as 1 to n^3; while the resistance to those ruptures increases only as 1 to n^2.

The structure, therefore, will have so much less firmness than the model, as n is greater.

If W be the greatest weight which one of the beams of the model can bear, and w the weight or stress which it actually sustains, then the limit of n will be $n = \frac{W}{w}$.

Prop. 2. The side of a model being to the corresponding side of the structure as 1 to n, the stress which tends to crush the parts by compression, increases from the smaller to the greater scale, as 1 to n^3, while the resistance increases only in the ratio of 1 to n.

Hence, if W were the greatest load which a modular wall, or column, could carry, and w the weight with which it is actually loaded; then the greatest limit of increased dimensions would be found from the expression $n = \sqrt{\frac{W}{w}}$.

If, retaining the length or height nh, and the breadth nl, we wished to give to the solid such a thickness xt, as that it should not break in consequence of its increased dimensions, we should have $x = n^2 \sqrt{\frac{w}{W}}$.

In the case of a pilaster with a square base, or of a cylindrical column, if the dimension of the model were d, and of the largest pillar, which should not crush with its own weight when n times as high, xd, we should have

$$x = n\sqrt[3]{\frac{n^2 w}{W}}.$$

These theorems will often find their application in the profession of an engineer, whether civil or military.

END OF VOL. II.

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